Quantum Mechanics

September 8, 2003

Work 4 of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

1. Consider a Hamiltonian of the form

$$H_0 = \frac{\mathbf{p}^2}{2m} + V_0(r).$$

$V_0(r)$ is not specified. We only know that it is rotational invariant and that there are a few bound states.

a. What can you tell about angular momentum, parity, and degeneracy of the ground state? (1 point)

b. What possibilities exist for angular momentum, parity, and degeneracy of the first excited state? (1 point)

c. What possibilities exist for angular momentum, parity, and degeneracy of the second excited state? (1 point)

d. Suppose we add a small perturbation of the form

$$V_1 = \varepsilon r,$$

where ε is a small constant. What happens to the ground state energy (up, down unchanged, impossible to tell)? What is the dependence of the ground state energy on ε (linear, quadratic, impossible to tell)? (1 point)

e. Suppose we add instead a small perturbation of the form

$$V_1 = \varepsilon z,$$

what happens to the ground state energy (up, down unchanged, impossible to tell)? What is the dependence of the ground state energy on ε (linear, quadratic, impossible to tell)? (2 points)

f. Same as question e., but for the first excited state. Note that you have to discuss different cases, dependent on the quantum numbers of the first excited state! (4 points)

Problem 2

Consider the one-dimensional single-particle Hamiltonian

$$H = H_0 + H'(t),$$

where

$$H_0 = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2$$
$$H'(t) = \delta x^3 \cos\omega_0 t f(t)$$

and where

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < \tau \\ 0 & \tau < t \end{cases} .$$

a) What quantity must δ be much smaller than, so that we can use perturbation theory?

b) Assuming that a particle is in the ground state of H_0 at $t = 0$, what is the probability of finding it in the n^{th} excited state of H_0 for $t > \tau$? Calculate to first order in δ.

Problem 3

Symmetries are fundamental concepts to our understanding of nature. For example, Newtonian Mechanics is invariant with respect to Galilean transformations, special relativity and QED, on the other hand, are Lorentz-invariant. This raise the question: To which of the two symmetry classes belongs quantum mechanics?

1) Show explicitly that the time-dependent Schrödinger equation for a free particle is invariant with respect to Galilean transformations. (Hints: Write down the time-dependent Schrödinger equation for a free particle, show that the general solution for the free particle can be written as a plane wave. Relate the coordinates in the laboratory frame to the coordinates in the moving frame. Write down the wave functions in both coordinate frames; How are position, momentum and energy related in both systems? Finally show that the two wave function are identical except for a phase factor).

2) Can this result be generalized to interacting systems? (Hint: Do plane waves form a complete basis of a Hilbert space?)

Problem 4

A particle of mass m is contained in a one-dimensional impenetrable box extending from $x = -\frac{1}{2}L$ to $x = \frac{1}{2}L$. The particle is in its ground state.

a) Find the eigenfunctions of the ground state and the first excited state.

b) The walls of the box are moved outward instantaneously to form a box extending from $-L \geq x \geq L$. Calculate the probability that the particle will stay in the ground state during this sudden expansion.

c) Calculate the probability that the particle jumps from the initial ground state to the first excited final state.

Problem 5

Consider the scattering of a spinless hadron from a central potential. We will be interested in this system as a model for the scattering of a neutral pion from a nucleon which has the interesting property that the scattering amplitude vanishes at threshold while it is negative and proportional to the kinetic energy for small positive energies.

The Klein-Gordon equation may be appropriate as a wave equation to describe this interaction. The free KG equation is obtained from the relativistic relationship between energy and momentum. (Units are such that $\hbar$ and c are unity.)

$$\omega^2 - p^2 = m^2 \rightarrow (\omega^2 + \nabla^2)\psi = m^2\psi$$

The introduction of a potential may be done in two ways. One of them is to replace the 4-vector

$$\omega, \mathbf{p} \rightarrow \omega - V_\omega, \mathbf{p} - \mathbf{A},$$

as would be the case for an electro-magnetic interaction (we might set $\mathbf{A} = 0$ in the static limit). Another possibility is the replacement

$$m \rightarrow m - V_m$$

for a purely scalar potential. Consider the case in which both potentials are introduced together in the static limit.

Take for the form of the potentials

$$V_\omega = 0 \ \text{ if } r > R; \quad V_\omega = V \ \text{ if } r \leq R$$

$$V_m = 0 \ \text{ if } r > R; \quad V_m = S \ \text{ if } r \leq R$$

$$\mathbf{A} = 0.$$

1) Solve for the s-wave scattering amplitude.

2) Show, by expanding the result found in part 1 around threshold, that, for $S = V > 0$, the desired behavior of the amplitude is obtained.