Quantum Mechanics

September 7, 2007

Work 4 of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Consider the elastic scattering of a low energy particle of mass, μ, and energy, E, from the "hard sphere" potential:

$$\begin{aligned} V(r) &= \infty, \quad r \leq a \\ &= 0, \quad r > a \end{aligned}$$

(a) Derive an expression for $\tan \delta_\ell$ where δ_ℓ is the phase shift in partial wave ℓ and show that for $\ell = 0$,

$$\tan \delta_0 = -\tan ka$$

where $k^2 = 2\mu E/\hbar^2$.

(b) Show that the low-energy limit of the elastic cross section, σ_0 is given by

$$\lim_{k \to 0} \sigma_0 = 4\pi a^2.$$

(c) Obtain an expression for the s-wave contribution to the scattering amplitude, $f(0)$, and verify the optical theorem

$$\sigma_{Total} = \frac{4\pi}{k} Im\ f(0)$$

for $ka << 1$.

Problem 2

Consider the Schrödinger equation in one dimension for a particle of mass m bound in a linear potential

$$V(x) = \frac{\hbar^2 a^3}{2m}|x|.$$

Calculate the variational estimate of the ground-state energy using the trial wave function

$$\psi_T(x) = e^{-\frac{1}{2}\alpha^2 x^2}.$$

Use α as the variational parameter.

Problem 3

The force constant k of the C-O bond in carbon monoxide is $1.87 \cdot 10^6 g/s^2$. Assuming that the vibrational motion of CO is purely harmonic and using the reduced mass $\mu = 6.875\,\text{amu}$:

a. Calculate the spacing between vibrational energy levels in this molecule in units of ergs and cm^{-1}.

b. Calculate the uncertainty $\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ in the internuclear distance x in this molecule, assuming it is in its ground vibrational level.

c. Under what circumstances (i.e., large or small values of k; large or small values of μ) is the uncertainty in internuclear distance large? Comment on any relationship between this observation and the fact that helium remains a liquid down to absolute zero.

Problem 4

For scattering of slow neutrons off protons, calculate the probability that the neutron will flip its spin, as a function of the singlet and triplet scattering amplitudes a_0 and a_1, when the beam and target spins are opposite.
Hint: For slow neutrons you only need to consider the lowest partial wave.

Problem 5

This is a multiple choice test. Do **NOT** show your work! Note question V on the back.

I. **T**rue or **F**alse:
Because of the symmetrization postulate, the three-particle wavefunction for three identical (spin 1/2) fermions, all with spin up, can always be written in the form
$\psi(x_1,x_2,x_3) = \psi_1(x_1)\psi_2(x_2)\psi_3(x_3) - \psi_1(x_1)\psi_2(x_3)\psi_3(x_2) - \psi_1(x_2)\psi_2(x_1)\psi_3(x_3)$
$-\psi_1(x_3)\psi_2(x_2)\psi_3(x_1) + \psi_1(x_3)\psi_2(x_1)\psi_3(x_2) + \psi_1(x_2)\psi_2(x_3)\psi_3(x_1)$ with suitable ψ_1, ψ_2, and ψ_3. true ◯ false ◯

II Consider 2 indistinguishable spinless bosons each bound in the same central potential $V(r)$, i.e.

$$V = V(r_1) + V(r_2).$$

The bosons are not interacting with each other, only with the potential V.

a.) What is the degeneracy of the system in its ground state?

b.) What is the degeneracy of the system in its first excited state?

III. Consider a particle subject to the Hamiltonian

$$H = \frac{1}{2m}\vec{p}^2 + V(r) + \gamma(x^2+z^2) + \gamma\vec{r}\cdot\vec{S},$$

where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the spin and γ is a constant. Check all that apply.

a. $\vec{L}\cdot\vec{S}$ is conserved ◯
b. $\vec{r}\cdot\vec{S}$ is conserved ◯
c. $L_x + S_x$ is conserved ◯
d. $L_y + S_y$ is conserved ◯
e. $L_z + S_z$ is conserved ◯
f. $\vec{L}^2$ is conserved ◯
g. $\vec{S}^2$ is conserved ◯
h. parity is conserved ◯
i. energy is conserved ◯
j. none of the above ◯

Note: You get 1 point for every correct checkmark and $-1/2$ for every wrong checkmark. However, the total points on this question cannot be less than 0.

IV. For the potential in the previous problem, check all correct statements

a. The ground state has $\vec{L}^2 = 0$ ◯
b. The ground state has $L_x = 0$ ◯
c. The ground state has $L_y = 0$ ◯
d. The ground state has $L_z = 0$ ◯
e. The ground state has positive parity ◯

Note: You get 1 point for every correct checkmark and $-1/2$ for every wrong checkmark. However, the total points on this question cannot be less than 0.

V. Two indistinguishable spin 1/2 fermions (both with spin up) interact through the Hamiltonian

$$H = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} + \frac{1}{2}\mu\omega^2 \left(\vec{r}_1 - \vec{r}_2\right)^2 .$$

with $\mu = m/2$.

a.) What is the energy of the ground state?

b.) What is the degeneracy of the ground state?

c.) What is the energy of the first excited state?

Note: consider only 'internal' excitations (total momentum zero)!