# Quantum Mechanics

February 2, 2007

Work 4 of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

## Problem 1

An electron is in a magnetic field of the form $\vec{B} = \vec{B}_0 + \vec{B}_1$, where $\vec{B}_0 = B_0\hat{z}$ and $\vec{B}_1 = \frac{B_1}{2}[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}]$. The goal in the following is to find the probability for the spin of the electron to point in the $-z$ direction at time $t$, given that it points in the $+z$ direction at $t = 0$.

(A) Write down the time-dependent Schrödinger equation for this problem.

(B) Since we do not know whether any of the magnetic fields are small, we cannot apply perturbation theory. However, the problem can be solved analytically. Use the following ansatz for the wave function,

$$\psi(t) = \begin{pmatrix} a\exp(i\omega_a t) \\ b\exp(i\omega_b t) \end{pmatrix}$$

and solve for $a$, $b$, $\omega_a$ and $\omega_b$.

(C) Assuming resonance, $\omega = 2\Omega$ (for notation, cf. below), show that the probability to find the electron in the $-z$ state is $P(t) = \sin^2(\beta t)$.

(D) Show that the (Rabi) resonance frequency is given by $\omega_R = 2\beta$.

Some hints and abbreviations that you may find useful:

1. Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \vec{S} = \frac{\hbar}{2}\vec{\sigma}\ .$$

2. Bohr magneton:

$$\mu_B = \frac{e\hbar}{2m_e c}\ .$$

3. Notation:

$$\beta \equiv \frac{\mu_B B_1}{2\hbar}\ , \quad \Omega \equiv \frac{B_0 \mu_B}{\hbar} \quad \text{and} \quad \Delta = \sqrt{\beta^2 - \left(\Omega - \frac{\omega}{2}\right)^2}\ .$$

4. If a wave function is normalized at some instant in time, it remains normalized at other times.

5. You may approximate the gyromagnetic ratio of the electron as $g = 2$.

# Problem 2

The following Hamiltonian matrix has been constructed using an orthonormal basis,

$$H = H^0 + H' \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$$

where $c$ is a constant.

(A) Find the exact eigenvalues of $H$.

(B) Use perturbation theory around the unperturbed Hamiltonian $H^0$ to determine the eigenvalues correct to second order.

(C) Compare the results of steps (A) and (B).

# Problem 3

For two identical fermions interacting via a short-range potential $U(r)$, find the effective scattering cross section in the low-energy limit and discuss the differences when the total spin of the two particles is even or odd in energy dependence and angular dependence of the cross section.

# Problem 4

Answer all 5 parts and use diagrams where appropriate:

(A) What is the Beer-Lambert Law? Show it diagrammatically and describe it mathematically.

(B) What is Raman spectroscopy and what form of scattering does it look at?

(C) What is the difference between Raman scattering, resonance Raman scattering, Stokes and anti-Stokes shifts and Rayleigh scattering?

(D) Describe (with diagrams) the difference between electronic and vibrational transitions.

(E) Describe what is meant by the density of states in a material.

# Problem 5

Answer all three questions.

1. **Momentum Operator:**
   Consider a particle such that the expectation value of its momentum is non-zero. Can the position-space wave function $\psi(\vec{r})$ be chosen to be real? Justify your answer (let it be noted that correct answers with incorrect justification will not count). *(2 points)*

2. **Addition of Spins:**
   Three spin-half particles interact with each other such that the Hamiltonian of the system is given by
   $$H = C\left(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3\right),$$
   where $C$ is a constant. Find the energy levels of this system and their degeneracy. *(5 points)*

3. **WKB Approximation:**
   In the WKB approximation, find the allowed energy levels that a ball of mass $m$, bouncing due to gravity on a perfectly reflecting surface, can have. You can use the fact that for this problem the WKB approximation gives
   $$\oint p dq = \left(n - \frac{1}{4}\right) 2\pi\hbar, \qquad n = 1, 2, 3, ...$$
   where the integral is over a periodic full path. You may leave your result as an integral that could be solved to yield the energy levels. *(3 points)*