Electromagnetism

September 22, 2008

Work 4 of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Given a circular ring of radius a made of wire of resistivity ρ_0 and radius $b \ll a$. An oscillating current $I(t) = I_0 \sin \omega t$ is maintained in the wire, with $\omega a/c \ll 1$.

a. Express the electrical resistance R of the ring in terms of ρ_0, a and b. Suppose $\rho_0 = 1.7 \times 10^{-6} \Omega\,\text{cm}$, which is about the resistivity of copper, and $a = 10\,\text{cm}$, $b = 10^{-1}\,\text{cm}$. Obtain a numerical value for R in Ohms.

b. Obtain an expression for the average thermal power P_{th} radiated in steady state, equal to the average resistive power dissipated. If $I_0 = 1\,\text{A}$, evaluate this average power in Watts.

c. The ring of oscillating currents acts as a magnetic dipole antenna. In Gaussian units, the magnetic dipole moment is defined by

$$\vec{\mu} = \frac{1}{2c} \int d^3r\, \vec{r} \times \vec{\jmath}\,.$$

Let the ring be in the $x - y$ plane. Show that $\vec{\mu}$ is in the $+z$ or $-z$ direction with magnitude $\mu(t) = \pi a^2 I/c$ for $b \ll a$. Hint: Let $\vec{\jmath} = \vec{e}_\varphi I \delta(z) \delta(\rho - a)$, with ρ the radial cylindrical coordinate and $\vec{e}_\varphi$ the azimuthal unit vector.

d. The average power radiated via magnetic dipole radiation due to the oscillating current is given by

$$P_{dip} = \frac{2}{3c^3} \left\langle (\ddot{\mu})^2 \right\rangle \,.$$

i. Using $I_0 = 1\,\text{A} = 3 \times 10^9$ stat A, $\omega = 10^6\, s^{-1}$, obtain a value for P_{dip}, and compare with P_{th}. Note: $1\,\text{erg}\, s^{-1} = 10^{-7}$ Watts.

ii. Show by dimensional analysis that the expression given for P_{dip} is correct to within a dimensionless constant. Hint: In Gaussian units, the dimension of μ is $[QL]$, charge times length, and Q^2/L is an energy, so the dimension of Q^2 is what in terms of $[M, L, T]$?

Problem 2

The entire region below the $x - y$ plane, $z < 0$, is filled by a superconductor, and an infinite straight line with current $\vec{I} = I\vec{e}_x$ is placed at a distance d above the $x - y$ plane. Find the magnetic field $\vec{B}$ and the induced current $\vec{K}$. Sketch the field lines.

Problem 3

Consider the Poisson equation for the electrostatic potential,

$$\Delta\Phi(\vec{r}) = \rho(\vec{r}) \ , \tag{1}$$

with the boundary condition that $\Phi \to 0$ as the distance from the origin goes to infinity, for a charged circular disc centered on the origin, $\rho(\vec{r}) = \theta(R-r)\delta(z)$ in cylindrical coordinates (r, φ, z).

1. Using the appropriate Green's function, calculate Φ on the positive z-axis with $z > R$ directly. Show that the result is

$$\Phi(r = 0, \varphi, z > R) = -\frac{1}{2}\left[\sqrt{z^2 + R^2} - z\right] \tag{2}$$

2. Converting to spherical coordinates (r, ϑ, φ), expand $\Phi(r > R, \vartheta = 0, \varphi)$ in a power series in $1/r$.

3. Insert the ansatz

$$\Phi(r > R, \vartheta, \varphi) = \sum_{l=0}^{\infty} a_l \frac{1}{r^{l+1}} P_l(\cos\vartheta) \tag{3}$$

   into eq. (1) for $r > R$ and show that it solves that equation for arbitrary constant coefficients a_l.

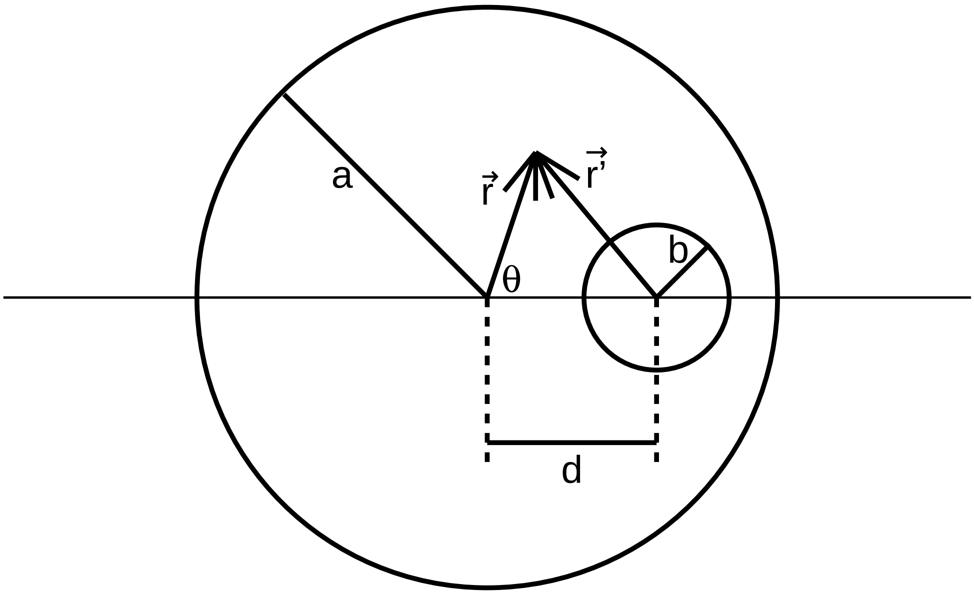
4. Fix the coefficients a_l using the behavior you derived in (b) at $\vartheta = 0$.

Problem 4

A region $0 \leq x \leq L$ is filled with a material of conductivity $\sigma \equiv \sigma(x) = a/x$, where a is a constant. The plane $x = 0$ is grounded (held at zero potential), and the plane at $x = L$ is held at constant potential $V_0 > 0$. Consider the steady-state solution, where all quantities are time-independent. Find the current density $\vec{J}$, the electric field $\vec{E}$, and the charge density ρ in the entire region $0 \leq x \leq L$.

Problem 5

A cylinder of radius a and uniform charge density ρ has a cylindrical hole of radius b bored into it, with the axis of the hole and the axis of the original cylinder parallel to each other. The distance between the axes is d. In the following, disregard boundary effects from the ends of the cylinder.



a. Give the electric field $\vec{E}$ inside the hole.

b. The electric field becomes zero somewhere in the cylinder. Where?