Quantum Mechanics

August 28, 2009

Work 4 of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Given a point particle, mass m, energy $E > 0$, moving in one dimension, x, in a potential energy $V(x) = \infty, x < 0$; $V(x) = W_0\delta(x-b)$, where $W_0 > 0,\ b > 0$ are constants.

a) Show that the Schrödinger equation for this problem is

$$u''(x) + k^2u(x) = \Gamma\delta(x-b)u(x),\ \text{where } k^2 \equiv 2mE/\hbar^2,\quad \Gamma = 2mW_0/\hbar^2$$

and the wavefunction $\psi(x,t) = u(x)e^{-iEt/\hbar}$.

b) Show that one boundary condition (BC) is $u'(b^+) - u'(b^-) = \Gamma u(b)$, also give a reason why the other BC is $u(b^+) = u(b^-) = u(b)$, where $b^\pm = b \pm \epsilon,\ \epsilon$ infinitesimal > 0.

c) Give the reasons: i) why $u(x) = 0,\ x \leq 0$; ii) in region 1, $0 \leq x < b$, why $u(x)$ may be written as $u_1(x) = A\ \sin kx$; and iii) in region 2, $x > b$, why it may be written $u_2(x) = A_{in}e^{-ik(x-b)} + A_re^{ik(x-b)}$, where we may choose $A_{in} = 1$, and A_r is a constant. Hint: In what direction is the wavefunction $e^{-ikx-iEt/\hbar}$ traveling?

d) Apply the BC's to these forms and show that
i) $A = -2i/D,\ D \equiv \cos(kb) + (\gamma - i)\sin(kb),\ \gamma = \Gamma/k$
ii) $A_r = -D^*/D$.

e) Consider $\gamma >> 1$. Show that then $|A|^2$ is a maximum ≈ 4 for $kb \approx n\pi$, and a minimum $\approx 4/\gamma^2$ for $kb \approx (n+\frac{1}{2})\pi,\ n = 1,2,3,\ldots$

f) Explain why these results make sense. In particular: i) Give a physical rationale for $|A|^2$ being small except when $kb \to n\pi$; and ii) What is the value of the reflectivity $|A_r|^2$ in general, and what is the physical reason that it must have this value.

Problem 2

Solve the following two quantum mechanical eigenvalue problems:

a. *(4 points)* Determine the energy eigenvalues for the one-dimensional Schrödinger equation with

$$V(x) = \begin{cases} -e^2/x & \text{for } x > 0 \\ +\infty & \text{for } x < 0 \end{cases}$$

Hint: The spectrum of the three-dimensional hydrogen problem with $V(r) = -e^2/r$ is $E_n = -me^4/(2\hbar^2 n^2)$ with $n = 1, 2, 3, \ldots$

(1 point) What is the degeneracy of each eigenvalue?

b. *(5 points)* Use the Bohr-Sommerfeld quantization rule to calculate the energy spectrum of an elastic ball which is bouncing vertically.

Problem 3

If one considers systems capable of emitting particles of half-integral spin, one encounters operators U obeying the commutation relations

$$
\begin{aligned}
[U, J_z] &= \frac{1}{2}U && (1) \\
\left[\left[U, J^2\right], J^2\right] &= \frac{1}{2}(UJ^2 + J^2U) + \frac{3}{16}U \ , && (2)
\end{aligned}
$$

where $\vec{J}$ is the angular momentum of the emitting system. Find selection rules following from (1) and (2), in a matrix representation which makes J_z and J^2 diagonal (eigenvalues m and $j(j+1)$, respectively). In other words, what matrix elements $\langle m'j'|U|mj\rangle$ can be nonzero? (*Hint:* Let $X_j = j(j+1)$.)

Problem 4

Einstein told us that the relativistic expression for the energy of a particle having rest mass m and momentum p is $E^2 = m^2c^4 + p^2c^2$.

a. Derive an expression for the relativistic kinetic energy operator which contains terms correct through one order higher than the "ordinary"

$$E = mc^2 + \frac{p^2}{2m}$$

b. Treating the additional term as a perturbation, determine the first-order perturbation theory estimate of the energy for the $1s$ level of a hydrogen-like atom (general Z). Show the Z-dependence of the result.

c. For what value of Z does the first-order relativistic correction amount to 10% of the unperturbed nonrelativistic $1s$ energy?

Problem 5

Consider the bound-state solution of the Schrödinger equation in three dimensions for zero angular momentum and a non-singular central potential.

a) Assume that the first derivative of the wave function with respect to the spherical coordinate, r, is non-zero at the origin. Sketch the wave function as a function of z for $x = y = 0$ around $z = 0$. Include both the regions $z < 0$ and $z > 0$. Discuss the problem with the kinetic energy operator at the origin.

b) For the solution in a potential which can be expanded in an infinite series in r

$$-\frac{2m}{\hbar^2}V(r) = \sum_{n=0}^{\infty} v_n r^n$$

show that this problem does not exist. You may find it easier to work with the function which is r times the wave function, $U(r) = r\psi(r)$ since the transformed Schrödinger equation has no first derivative term.

c) Under what conditions is the third derivative of the wave function with respect to r zero.