Electromagnetism

September 20, 2010

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

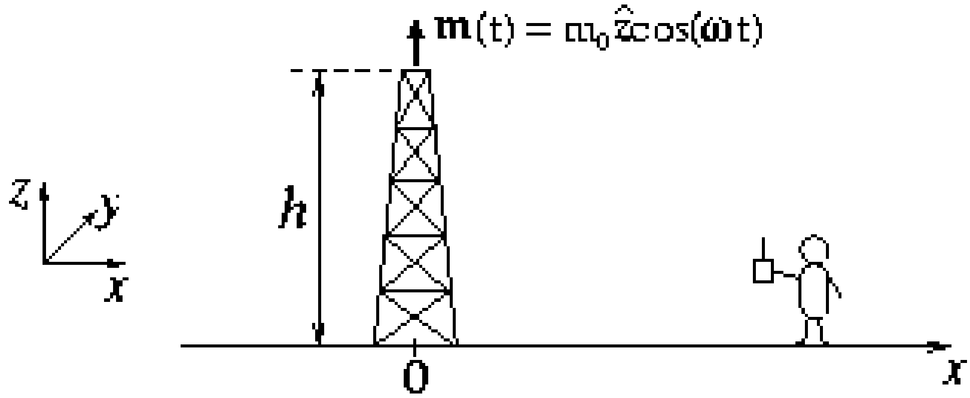
Problem 1

A metal containing **free** charges of mass m and charge q with a charge density n per unit volume is exposed to an electric field $E(t) = E_0 \exp(-i\omega t)$ with amplitude E_0 and angular frequency ω. Assume that the frictional force on moving charges in the metal is proportional to the velocity of the charges with damping constant b.

1. What is the equation of motion for one of the charges in this metal?
2. How would you describe this type of motion in two words?
3. Determine the position of the charge as a function of time.
4. What is the (complex) dielectric constant $\epsilon(\omega)$ as a function of angular frequency ω for this metal with charge density n? Write your result in terms of the plasma frequency $\omega_P^2 = nq^2/m\epsilon_0$ and the damping rate $\gamma = b/m$.
   Hint: Remember that the dielectric polarization is defined as the dipole moment per unit volume.
5. If we neglect damping, what happens at the plasma frequency?
6. Plot the real and imaginary parts of the dielectric function in units of $x = \omega/\omega_P$ from 0.2 to 2. Assume that $\gamma = 0.1\omega_P$.
7. In one of the "Star Trek" movies, the Chief Engineer "invents" transparent aluminum. Why is this impossible based on our model? How would you minimize the absorption of a metal at a given wavelength (say, 633 nm) by changing the material parameters of the metal?

Problem 2

A radio transmitting tower rises to height $h = 100\,\text{m}$ above flat horizontal ground. A small vertical magnetic dipole antenna is placed at the top of the tower. The antenna operates at a frequency of $f = 100\,\text{MHz}$. The total time-averaged power radiated by the antenna is $P = 10\,\text{kW}$.



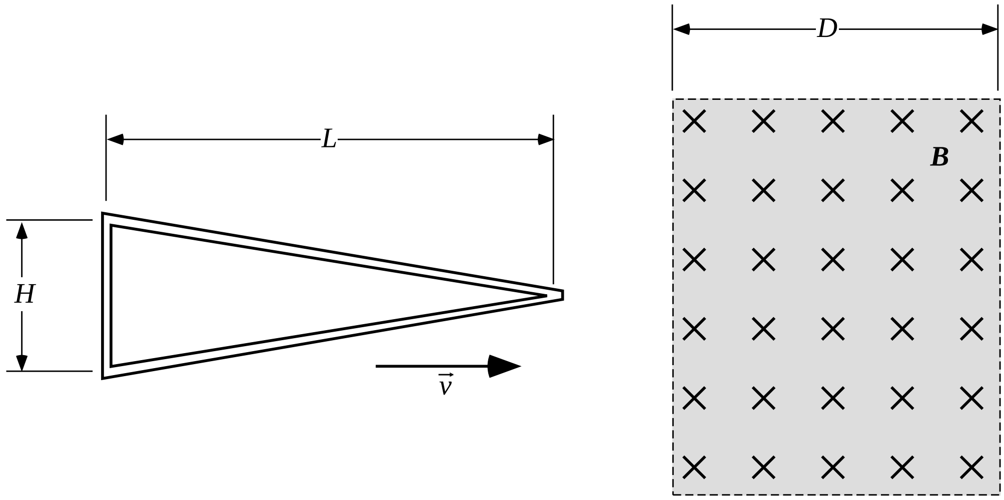
(a) Calculate the intensity of electromagnetic radiation at the ground level as a function of the distance from the base of the tower.

(b) Find the *maximum* intensity of radiation at the ground level.

(c) If you have a portable radio receiver, how far from the base of the tower should you stand to get the strongest radio signal?

Problem 3

A uniform magnetic filed B points into the page in a region of width D. The magnetic field is zero everywhere else (ignore fringing fields). An external agent pushes a wire loop shaped into an isosceles triangle of length L (note that $L > D$) and height H such that the triangle moves to the right with a constant speed v. The mass of the loop is m, and the resistance of the wire is R.



The point of the triangular loop enters the magnetic field at time $t = 0$. Graph the current induced in the loop as a function of time. Label the graph at all points at which the shape of the graph changes with the time and the current values expressed in terms of the given variables. Be sure to make clear which times the current is clockwise and which times it is counterclockwise.

Graph also the force applied by the external agent as a function of time, again labeling all important force values and times in terms of given variables.

Problem 4

Self-force on a static charge-current distribution.
Given a time-independent charge density $\rho(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$ that are localized, i.e., equal to zero outside some spherical surface of radius a centered at the origin.

a) Show that $\int d^3r \mathbf{j}(\mathbf{r}) = 0$, where the integral extends over all space.
Hint: Investigate $\int d^3r \partial_k(j_k x_i)$ and use the divergence theorem and continuity, $\nabla \cdot \mathbf{j} = \partial_k j_k = 0$. Note that summation notation is being used.

b) Show that $\int d^3r(x_j j_k + x_k j_j) = 0$.
Hint: Consider $\int d^3r \partial_i(j_i x_j x_k) =?$ and use the (generalized) divergence theorem and continuity.

c) Using the results of a) and b), show that for $r >> a$, the leading terms in the potentials $\phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ in inverse powers of r are $\phi(\mathbf{r}) = q/r$, $\mathbf{A}(\mathbf{r}) = (\boldsymbol{\mu} \times \mathbf{r})/r^3$.
Hint: Start from the general expressions $q = \int d^3r' \rho(\mathbf{r}')$, $\boldsymbol{\mu} = \frac{1}{2c} \int d^3r' \mathbf{r}' \times \mathbf{j}(\mathbf{r}')$, $\phi = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$, $\mathbf{A} = \int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{c|\mathbf{r}-\mathbf{r}'|}$, and use $1/|\mathbf{r} - \mathbf{r}'| = r^{-1} + \hat{\mathbf{r}} \cdot \mathbf{r}'/r^2$ for $r >> r'$. Note: We use Gaussian units.

d) Show that the self-Lorentz force on the charge-current distribution is zero: $\mathbf{F}_s = \int d^3r(\rho \mathbf{E} + c^{-1}\mathbf{j} \times \mathbf{B}) = 0$, where $\mathbf{E}$ and $\mathbf{B}$ are the fields due to ρ and $\mathbf{j}$.
Hint: Use the Maxwell equations $\rho = \frac{1}{4\pi} \nabla \cdot \mathbf{E}$, $\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}$, vector identities, and the (generalized) divergence theorem, as well as the results of part (c), with $\mathbf{E} = -\nabla\phi$, $\mathbf{B} = \nabla \times \mathbf{A}$.
The self-force had better be zero: A particle at rest can't accelerate itself! Be nice if it could: That would solve all our energy problems. But ....

Problem 5

Consider a dielectric sphere of radius R centered at the origin carrying a uniform polarization P in z-direction, $\vec{P} = P\vec{e}_z$. Determine the bound charge density and the electric field, both inside and outside the sphere.