Quantum Mechanics

September 17, 2010

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

1. Consider a harmonic oscillator in 11 dimensions

$$H = -\frac{\hbar^2}{2m}\sum_{k=1}^{11}\frac{d^2}{dx_k^2} + \frac{1}{2}m\omega^2\sum_{k=1}^{11}x_k^2$$

   a. What is the spectrum of H?
   (note: when you write $E_n = ...$, please be specific about the values of n)

   b. What is the degeneracy of the 1^{st} excited state?

   c. What is the degeneracy of the 2^{nd} excited state?

2. consider the two Hamiltonians

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{c}{x^2+a^2} \qquad \text{and} \qquad \tilde{H} = -\frac{1}{2}\hbar^2\frac{d^2}{dx^2} - \frac{c}{x^2+a^2} - \frac{c}{(x-R)^2+a^2}.$$

   a. What condition does c have to satisfy so that the H has a bound state (ground state energy $E_0 < 0$)? Briefly explain your answer.

   b. Assuming the parameters in this problem are such that both H and $\tilde{H}$ have bound states (with ground state energies E_0 and $\tilde{E}_0$ respectively). Which of the following statement is true

   O $0 > E_0 > \tilde{E}_0$

   O $0 > E_0 = \tilde{E}_0$

   O $E_0 < \tilde{E}_0 < 0$

   O Impossible to compare without further information about the numerical values of a, c, m, R

   Briefly explain your answer.

Problem 2

Consider a particle of mass m moving in one dimension in a potential given by

$$V = V_0 \text{ for } x < 0; \quad V_0 > 0$$

$$V = 0 \text{ for } 0 \leq x \leq a$$

$$V = \infty \text{ for } x > a$$

a) First consider the limit as $V_0 \longrightarrow \infty$. Find the energies of the allowed states, E_n, under these conditions.

Now take the case for V_0 finite but very large. Consider only values of n for which $E_n/V_0 << 1$.

b) Find the (transcendental) equation which will allow the calculation of these new bound state energies, E'_n, (in terms of momenta, k'_n where $E'_n = \frac{\hbar^2}{2m}k'^2_n$).
c) Find the transcendental equation for the shift, $\delta_n \equiv k'_n - k_n$. in momentum from the values found in part a,
d) Expand the result in part c to first order in δ_n to get an approximate solution for the energy shift,

$$\Delta E_n \equiv E'_n - E_n.$$

e) Show that the fractional energy shift, $\Delta E_n/E_n$ is independent of n. Find this fractional shift.

Problem 3

Two identical non-interacting particles are in an isotropic harmonic potential. Show that the degeneracies of the three lowest energy levels are:

(a) 1, 12, 39, if the particles have spin $\frac{1}{2}$

(b) 6, 27, 99, if the particles have spin 1.

Problem 4

The spinless, neutral particle K and its antiparticle $\bar{K}$ can convert into each other through a weak interaction: $K \rightleftarrows \bar{K}$ and therefore a state produced initially as $|\psi(0)>=|K>$ at $t=0$ will in general be a mixture of $|K>$ and $|\bar{K}>$ at time t. Furthermore, the linear combination

$$|K_S> \equiv \frac{1}{\sqrt{2}}\left[|K>+|\bar{K}>\right]$$

has a much shorter lifetime, τ_S, than the orthogonal state $|K_L>$: $\tau_S \ll \tau_L$. (Here, lifetime is defined in the usual sense: after a time t, measured in the particles rest frame, out of an initial number $N(0)$ there will remain $N(t)=N(0)e^{-t/\tau}$ particles.) The masses of the two states are m_S and m_L, respectively.

(a) Write expressions for the amplitudes $A_i(t)$, $i=S,L$ for the two states in terms of the so-called widths $\Gamma_i \equiv \hbar/\tau$ the masses m_i, and the initial amplitudes $A_i(0)$ at $t=0$, where t is measured at the corresponding state's rest frame. (*Hint:* What is the total energy of a particle in its rest frame?) (2 points)

(b) Consider a state that is produced at $t=0$ as pure K. Calculate the probability $P(\bar{K};t)$ that the state will be $\bar{K}$ after a time t. (3 points)

(c) After a time $t \gg \tau_S$, the $|K_S>$ state has essentially disappeared. What is the state $|\psi>$ at this point, in terms of $|K>$ and $|\bar{K}>$? (2 points)

(d) The states $|K>$ and $|\bar{K}>$, being those of a particle and its antiparticle, interact differently with matter. Define f and $\bar{f}$ to be the probabilities that a K or a $\bar{K}$ will be absorbed (and therefore disappear) if the state in **(c)** passes through a given amount of matter. Explain why the long-vanished, short-lived state $|K_S>$ will reappear under these conditions and calculate the content of K_S in this final state, *i.e.*, the probability that this state will be a K_S. (3 points)

Problem 5

Consider an electron constrained to move on the surface of a sphere of radius r_0. The Hamiltonian for such motion consists of a kinetic energy term only and is given by

$$H_0 = \frac{L^2}{2m_e r_0^2}$$

where L is the orbital angular momentum operator involving derivatives with respect to the spherical polar coordinates θ, ϕ. H_0 has the complete set of eigenfunctions

$$\psi_{lm}^{(0)} = Y_{l,m}(\theta, \phi)$$

a. Compute the energy levels of this system in the absence of any perturbation.

b. A uniform electric field is applied along the z-axis, introducing a perturbation

$$V = -e\epsilon z = -e\epsilon r_0 \cos\theta$$

where ϵ is the strength of the field. Evaluate the correction to the energy of the lowest level through second-order perturbation theory, using the identity

$$\begin{aligned}\cos\theta\, Y_{l,m}(\theta,\phi) &= \left(\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right)^{1/2} Y_{l+1,m}(\theta,\phi) \\ &\quad + \left(\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right)^{1/2} Y_{l-1,m}(\theta,\phi)\end{aligned}$$

Note that the identity enables you to utilize the orthonormality of the spherical harmonics.

c. The electric polarizability α gives the response of a molecule to an externally applied electric field and is defined by

$$\alpha = -\frac{\partial^2 E}{\partial \epsilon^2}$$

where E is the energy in the presence of the field and ϵ is the strength of the field. Calculate α for this system.

d. Use this problem as a model to estimate the polarizability of a hydrogen atom, where $r_0 = a_0 = 0.529$ Å, and a cesium atom, which has a single $6s$ electron with $r_0 \approx 2.60$ Å.