Statistical Mechanics

September 22, 2010

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and put your name on each sheet.

Problem 1

Consider a degenerate Fermi gas containing n particles per unit volume. Let ϵ_F be the Fermi energy of this gas.

(a) Derive an expression for the isothermal compressibility, $\kappa_T = -\frac{1}{V}(\frac{\partial V}{\partial P})_T$, of this gas at zero temperature.

(b) Derive an expression for the thermal expansion coefficient, $\alpha = \frac{1}{3V}(\frac{\partial V}{\partial T})_P$, of this gas. (*Hint: Use the chain rule for V, P, and T.*)

Problem 2

Paramagnetism of a stationary particle having charge q, mass m, and spin $\frac{1}{2}$, in a constant uniform magnetic field $\mathbf{B}$.

a) The conventional treatment assumes a magnetic moment $\boldsymbol{\mu} = (gq/2mc)\mathbf{S}$, where g is the "g-factor" ($g \approx 2$ for electrons), $\mathbf{S}$ is the spin angular momentum operator, with eigenvalues $\pm\frac{1}{2}\hbar$ for S_z, so that the eigenvalues of μ_z are

$$\mu_z = \pm gq\hbar/4mc \equiv \pm\mu_B \tag{1}$$

where μ_B is the Bohr magneton.

Let H_P be the Pauli Hamiltonian, $H_P = -\boldsymbol{\mu}\cdot\mathbf{B} = -\boldsymbol{\Omega}\cdot\mathbf{S}$ where $\boldsymbol{\Omega} = gq\mathbf{B}/2mc$. The eigenvalues of H_P are $E_\pm = \mp\hbar\Omega_z/2$ ($\mathbf{B}$ in the z-direction).

i) What is the (single particle) partition function Z for temperature T?

ii) The expectation $<\mu_z>$ is defined by

$$<\mu_z>= Z^{-1}[\mu_+ e^{-\beta E_+} + \mu_- e^{-\beta E_-}] \tag{2}$$

where $\beta \equiv (k_B T)^{-1}$ and k_B is Boltzmann's constant. Show that

$$<\mu_z>= \mu_B \tanh(\beta\hbar\Omega_z/2) \tag{3}$$

b) In the 1970's, R. Young investigated quantization of a spinning extended charged particle. For a spherically symmetric particle whose center is at rest, with moment of inertia I about its center, he showed that the Hamiltonian and the magnetic moment are

$$H = \frac{1}{2I}(\mathbf{S} - I\boldsymbol{\Omega})^2; \quad \boldsymbol{\mu} = (gq/2mc)(\mathbf{S} - I\boldsymbol{\Omega})$$

i) Using appropriate eigenvalues of μ_z and H in Eq. (2), show that now

$$<\mu_z>= \mu_B[\tanh(\beta\hbar\Omega_z/2) - 2I\Omega_z/\hbar].$$

ii) Consider room temperature, $k_B T = \frac{1}{40}$ eV, a large field $B_z = 10^4$ Gauss, and proton parameters, $q = e = 1.6\times10^{-12}$ esu, $mc^2 = 931$ MeV $\rightarrow m = 1.6\times10^{-24}$ g, and $I = ma^2$ with $a = 10^{-12}$ cm. Show that then $\beta\hbar\Omega_z/2 << 1$, so that $<\mu_z>= \frac{1}{2}\mu_B\beta\hbar\Omega_z[1 - 4I/\beta\hbar^2)]$, and $(4I/\beta\hbar^2) \approx 2.5\times10^{-7}$ which is certainly negligible. (1 eV $= 1.6\times10^{-12}$ erg, $c = 3\times10^{10}$ cm/s.)

Problem 3

A rotator with principal moments of inertia A, B and C is described by the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2A\sin^2\theta}\left((p_\phi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi\right)^2 \\ &\quad + \frac{1}{2B\sin^2\theta}\left((p_\phi - p_\psi\cos\phi)\sin\psi + p_\phi\sin\theta\cos\psi\right)^2 + \frac{1}{2C}p_\psi^2 \end{aligned}$$

where θ, ϕ and ψ are the Euler angles.

a. Derive the rotational partition function for this polyatomic molecule in the classical approximation, up to a constant. Hint: Carry out the integrations in the order p_θ, p_ϕ, $p_\psi, \ldots$

b. Derive the specific heat of this rotator; interpret your answer.