Electromagnetism

August 27, 2012

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

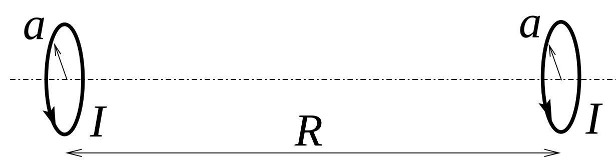
Consider the Laplace equation in two dimensions,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x, y) = 0 \ .$$

Find the complete set of solutions in **polar** coordinates, using standard separation ansatz techniques. Use this to fully determine the electrostatic potential when an infinitely long grounded conducting cylinder of radius R oriented along the z-axis is introduced into an initially uniform electric field $\vec{E} = E_0 \vec{e}_x$. Give also the surface charge density induced on the cylinder.

Problem 2

Two identical circular wire loops of radii a are located a distance R apart on a common axis perpendicular to their planes. The loops carry equal currents I flowing in the same direction. Calculate the force between the loops assuming that $R >> a$. Is this force attractive or repulsive?



Problem 3

An electron of charge e and mass m is bound by a linear restoring force with spring constant k. When the electron oscillates, the radiated power can be expressed as (Gaussian units):

$$p = \frac{2e^2}{3c^3}\dot{v}^2$$

where $\dot{v}$ is the acceleration of the electron and c is the speed of light. You may assume in the following problem that the electron oscillates with an angular frequency of

$$\omega = 10^5 \frac{rad}{s}$$

a) Consider the case where the radiation loss is solely due to the presence of a damping force, F_D . Consider the case where the energy loss is small as compared to the mechanical energy of the system. Derive an expression that relates F_D and $\dot{v}$.
b) Show that electron performs damped harmonic oscillations.
c) Compute the average mechanical energy of the electron.
d) How long does it take for the electron to lose half of its mechanical energy?
Hints:
Classical electron radius: $r_0 = \frac{e^2}{mc^2} = 2.82 \times 10^{-13}$m.
$\sqrt{1-x} \approx 1 - x/2$.

Problem 4

Given a plane mirror in the lab frame plane $z = \beta ct$. Let a plane electromagnetic wave pulse of finite duration T_i and (angular) frequency ω_i be normally incident on the mirror from vacuum on the left ($z < \beta ct$), resulting in a reflected pulse of duration T_F and frequency ω_F.

a) Show that $\frac{\omega_F}{\omega_i} = \frac{1-\beta}{1+\beta} = \frac{T_i}{T_F}$.

Hint: Let the incident wave go like $exp[i\omega_i(t - z/c)]$, the reflected wave like $exp[i\omega_F(t + z/c)]$, use the fact that the frequencies at the mirror surface must be the same. Also, what about the total number of oscillation cycles in each wave?

b) Let the pulse energies in the rest frame of the mirror be $\bar{E_F} = \bar{E_i}$, and the z-components of momenta be $\bar{P_i} = \bar{E_i}/c = -\bar{P_F}$, as must be the case by symmetry and the Maxwell eqns. Then apply the Lorentz transformations $z = \gamma(\bar{z} + \beta c\bar{t})$, $ct = \gamma(c\bar{t} + \beta\bar{z})$ to the four-momenta, and show that the pulse energies and momenta in the lab frame are related by $\frac{E_F}{E_i} = \frac{1-\beta}{1+\beta} = -\frac{P_F}{P_i}$, the same ratio for the energies as for the frequencies. Note that this is a classical result that can be obtained only if one uses pulses of finite duration instead of infinitely long plane waves.

c) Show that the work done on the mirror by the incident and reflected pulses, $W = E_i - E_F = \frac{2\beta}{1+\beta}E_i$, is also given by the dot product of the mirror velocity and the total impulse transferred to the mirror by the pulses, $W = \beta c(P_i - P_F)$, as must be the case.

d) Suppose the mirror is moving toward the source-detector, which is at rest in the lab frame, so that $\beta < 0$. Let $\beta = -0.9999$. Evaluate $\frac{\omega_F}{\omega_i} = \frac{E_F}{E_i}, \frac{T_F}{T_i}, \frac{W}{E_i}$. What is the meaning of $W < 0$.

e) Give argument(s) to support or refute the assertion that the usual results for non-normal incidence, including the results for work done on the mirror, should be obtained from the same model used here for normal incidence.

Problem 5

A dielectric with a (bound) charge density n per unit volume is under the influence of an electromagnetic field $E(t) = E_0 \exp(-i\omega t)$ with a wavelength much larger than the interatomic distance, resulting in forced oscillations of the charges. The amplitude E_0 of the electric field is small. The frictional force on the moving charge is proportional to the velocity of the charge with damping constant $b = \gamma m$. The restoring force on the charge is proportional to the displacement with constant k. You may assume that γ is small.

1. Draw a free-body diagram showing all forces acting on the charge.
2. What is the equation of motion for the charges?
3. Determine the position of the charge as a function of time.
4. What is the dielectric susceptibility $\chi(\omega)$ as a function of angular frequency ω for this dielectric with density n of the bound charges?
5. Plot the real and imaginary parts of $\chi(\omega)$ as a function of angular frequency ω. Discuss the high- and low-frequency limits and the location of the peak in the imaginary part of χ.