Quantum Mechanics

August 24, 2012

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Consider a quantum system which can exist in two states "A" and "B" and which can transition (decay) from A to B. Most commonly such a decay is considered to obey the "exponential decay law" where the probability of the system remaining in the state A at time t is given by

$$P(t) = e^{-\lambda t}$$

where λ is a constant which depends on the details of the system.
The solution of the quantum mechanical problem gives, in some cases, other possible decay laws. These predictions are rarely, if ever, seen in experiments. Suppose, however, a probability expression which is more general than that in the equation above (i.e. a non-exponential decay law) and consider the following experiment.

The state of the system is monitored continually at very short equal time intervals, δ. until, at the nth interval, it is observed to transition to state B. This process is repeated many times and the distribution of events as a function of n (or $t = n\delta$) is accumulated. What will be the decay law observed, $P_o(t)$ for any given fundamental law $P(t)$? That is, find the experimentally observed decay law as a function of the fundamental law.
Use your general expression to evaluate the observed law in the case of a fundamental exponential law.

Problem 2

Elastic scattering in Born approximation:

a.) Consider scattering of a particle of mass m off a potential with a discrete translational invariance, $V(\vec{r}+\vec{R}) = V(\vec{r})$, where $\vec{R}$ is a given, fixed vector. Show that, in first Born approximation, scattering can only occur in the directions defined by the condition $(\vec{k}-\vec{k}')\cdot\vec{R} = 2\pi n$ with integer n, where $\vec{k}$ is the incident wave vector and $\vec{k}'$ the wave vector of the scattered wave. Note that not all values of n need to give non-vanishing contributions.

b.) Consider now the specific potential

$$V = e^{-b(x^2+y^2)}e^{-az^2}\sin Qz$$

(which for $a \to 0$ becomes of the form discussed above; for finite a, the periodic array is cut off by the Gaussian envelope). Calculate the scattering amplitude in first Born approximation for arbitrary $\vec{k}$ and $\vec{k}'$. Check that the behavior is consistent with the one discussed in a.) for $a \to 0$.

c.) For the special case $b = a$, use your result from b.) to calculate the total cross section for scattering of an incoming wave propagating specifically in the positive z-direction.

Problem 3

Consider the solution of the one-dimensional Schrödinger equation with a potential

$$V(x) = 4\frac{\hbar^2}{2m}\alpha^2 x^2 \quad -\infty < x < \infty$$

a) Find the exact solution for the energy of the ground state. (2 pts)
b) Find the variational value for the energy of the ground state using the trial wave function

$$\psi_T(r) \propto \frac{1}{b^2 + x^2}$$

Use b as the variational parameter. How does the fractional error depend on α. (8 pts)
You might find useful the expression

$$\int_{-\infty}^{\infty} \frac{dx}{(b^2 + x^2)^n} = \frac{\pi 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{n-1} b^{2n-1} (n-1)!}$$

Problem 4

We are all familiar with the one-dimensional harmonic oscillator, which can be written in terms of creation and annihilation operators b^+ and b^-, obeying the commutation relations $[b^-, b^+] = 1$, $[b^+, b^+] = [b^-, b^-] = 0$: the Hamiltonian can be written as $H_B = \frac{1}{2}\omega \{b^+, b^-\}$ (we use units such that $\hbar = 1$). We can define an alternative harmonic oscillator in terms of creation and annihilation operators f^+ and f^- which obey similar *anti*commutation relations $\{f^-, f^+\} = 1$, $\{f^+, f^+\} = \{f^-, f^-\} = 0$ we write the Hamiltonian as $H_F = \frac{1}{2}\omega [f^+, f^-]$. Here $f^\pm$ act on the eigenstates of H_F the same way $b^\pm$ act on those of H_B:

$$\begin{aligned} f^+ |n_F\rangle &= \sqrt{n_F + 1}\, |n_F + 1\rangle \\ f^- |n_F\rangle &= \sqrt{n_F}\, |n_F - 1\rangle . \end{aligned}$$

Here n_F is the quantum number defining the eigenstates of the Hamiltonian, analogous to the quantum number (which we will write as n_B) that defines the states of the familiar harmonic oscillator.

(a) Show that $f^+ f^-$ is a number operator, counting the number of quanta n_F. (1 point)

(b) Find the spectrum of all possible values of n_F. (1 point)

(c) Based on the result of **(b)** and your knowledge of the corresponding spectrum for n_B, which you don't have to derive, justify why n_F describes fermionic quanta while n_B describes bosonic quanta. (1 point)

(d) Find the energy spectrum for H_F. How does this compare with the spectrum of the more familiar harmonic oscillator H_B? (Remember, we set $\hbar = 1$.) (2 points)

(e) Consider states $|n_B, n_F\rangle \equiv |n_B\rangle \otimes |n_F\rangle$, constructed by convolving solutions of the two oscillators; n_B is the number of bosonic quanta and n_F the number of fermionic quanta, write expressions for operators Q_+ and Q_- acting on such states, the former transforming a bosonic quantum into a fermionic one and the latter doing the opposite, in terms of $b^\pm$ and $f^\pm$. (1 point)

(f) Consider the Hamiltonian $H_S \equiv \omega \{Q_+, Q_-\}$. Show that it is invariant under the interchange of a fermionic and a bosonic quantum. (*Hint:* Consider appropriate commutators.) (2 points)

(g) Find the energy spectrum and the degeneracy of the eigenstates of H_S. (*Hint:* Write H_S in terms of H_B and H_F). (2 points)

Problem 5

1) **Scattering**: a) In Rutherford scattering (classically), will a smaller impact parameter lead to a larger or smaller scattering angle? (1 point)
b) True or False? If the scattering potential is spherically symmetric, so is the scattering amplitude. (1 point)
c) More or less? As the energy increases, does one need to include more or less partial waves in a partial wave expansion for the scattering amplitude? (1 point)
2) **Particle in a magnetic field**: a particle moves through a homogeneous magnetic field $(B, 0, 0)$ pointing in the x direction, described by the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 \quad \text{with} \quad \vec{A} = \frac{1}{2}\vec{r} \times \vec{B}.$$

Which of the following are conserved?
p_x
p_z
L_x
L_z
$\vec{L}^2$
where $\vec{L} = \vec{r} \times \vec{p}$. (2 points)
3. **Identical particles**: Would the wave function

$$\psi(\vec{r}_1, \vec{r}_2) = \left(\vec{r}_1^{\,2} + \vec{r}_2^{\,2} - \vec{r}_1 \cdot \vec{r}_2\right) e^{-(\vec{r}_1 - \vec{r}_2)^2}$$

be an "allowed" wave function for two identical spin 0 particles. Explain your reasoning. (2 points)
4. A particle moves inside a hollow spirally wound tube that is aligned along the $\hat{z}$ axis ($\hat{z}$ pointing to the right in the figure). The potential is zero inside the tube but has infinitely high walls so that the particle is trapped inside. A short segment of the infinitely long tube is shown in the figure. The pitch (displacement in the direction of the tube axis per winding) of the tube is 1 cm. Determine which linear combination of components of momentum and orbital angular momentum is conserved in this problem. (3 points)