Quantum Mechanics

August 18, 2016

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

The probability current

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

gives the probability that one particle per unit time will pass through a unit area normal to the direction of $\vec{J}$. A beam of particles with uniform velocity $\vec{v}$ enters a region of space where some of the particles are absorbed by atoms present there. This absorption can be represented by the introduction of a constant complex potential $V_r - iV_i$ into the wave equation. Show that the cross section per atom for absorption is $\sigma = 2V_i/(\hbar N v)$, where N is the number of absorbing atoms per unit volume.

Problem 2

A particle of mass M and charge q is constrained to move in a circle of radius R.

a.) If no forces other than those of constraints act on the particle, find its allowed energies and corresponding eigenstates. [*Hint: For a circle of radius R, the wave function depends only on the angular coordinate.*]

b.) A strong, uniform electric field, $\vec{E}$, oriented in the plane of the circle, is applied to the system. Find the pattern of low-lying eigenvalues and corresponding eigenstates under the assumption that $|\vec{E}|qR \gg \hbar^2/(2MR^2)$. [*Hint: For such large $|\vec{E}|$, the particle's wave function will be appreciable only for a small range of angles.*]

c.) If a uniform magnetic field $\vec{B}$ is applied perpendicular to the plane of the circle, find the resulting eigenvalues and eigenstates. Work this out for both the $\vec{E} = 0$ situation of part a.) as well as the $\vec{E} \neq 0$ case of part b.).

Problem 3

Two distinguishable particles, both of mass m, move and interact in three dimensions $[\vec{r}_i = (x_i, y_i, z_i),\ i = 1, 2]$ with the following Hamiltonian,

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{k}{2}(r_1^2 + r_2^2) + g(x_1x_2 + y_1y_2 - 2z_1z_2)\ ,$$

where $k > 0$.

What is the range of values of g for which a bound-state solution exists?

Problem 4

a.) A particle moves in a potential of the form ($\omega_1 \neq \omega_2$)

$$V(\vec{r}) = \frac{1}{2}m\omega_1^2 x^2 + \frac{1}{2}m\omega_2^2(y^2 + z^2)$$

Besides the Hamiltonian itself, list two other operators that commute with the Hamiltonian. (2 points)

b.) Two particles are subject to a Hamiltonian

$$H = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + \frac{1}{2}k(\vec{r}_1 - \vec{r}_2)^2$$

i.) Introducing center-of-mass $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ and relative $\vec{r} = \vec{r}_1 - \vec{r}_2$ coordinates, write down the Hamiltonian in terms of these variables. (4 points)

ii.) What are the energy eigenvalues for this Hamiltonian? If you introduce any quantum numbers, such as n, be specific about the values of n. (2 points)

iii.) If the two particles above are indistinguishable spin 0 particles, what are the allowed energy eigenvalues? Again, if you introduce any quantum numbers, such as n, be specific about the values of n. (2 points)

Problem 5

Suppose the Hamiltonian for two spin-1/2 particles is given by

$$H = A\vec{S}_1 \cdot \vec{S}_2 + B(S_{1z} + S_{2z}) \ .$$

a.) What are the energy levels and stationary states of the system?

b.) A perturbation

$$V = \Delta S_{1z}$$

is now added to the system. Calculate, in perturbation theory, the lowest non-zero correction to each energy eigenvalue.

c.) Solve the problem exactly, and verify that your exact solution agrees in the appropriate limit with the approximate one obtained in part b.).

d.) Suppose now that $B = \Delta = 0$, and the system has an initial state where particle 1 has spin up and particle 2 has spin down. What is the state of the system as a function of time? Is there ever a time when both particles point up (or both point down)?

Note: These two particles are not identical.