# Classical Mechanics

August 19, 2017

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

## Problem 1

a.) A body of mass $m$ is orbiting under the influence of a central force, described by a power law of the form:

$$\vec{F} = -Ar^n\hat{r}$$

Show that this can be written as:

$$\frac{1}{2}m\dot{r}^2 = E - U'(r)$$

with the effective potential

$$U'(r) = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}$$

where $J^2 = mr^2\dot{\theta}$. (2.5 points)

b.) A mass $m$ is attached to a pivot by a long light spring, which exerts a force $\vec{F} = -kr\hat{r}$, where $r$ is the distance of the mass from the pivot. The mass is set in motion and performs an orbit around the pivot. Neglecting gravity and friction, consider the $x$ and $y$ components of the force and acceleration separately to show that an orbit of the following form is permitted:

$$x = a\cos\omega t \qquad\qquad y = b\cos\omega t$$

(2 points)

c.) Show that this corresponds to an elliptical orbit. (0.5 points)

d.) Draw a diagram showing the form of $U'(r)$ as a function of $r$ for this force law. (1 point) e.) Consider the case where the spring has the force law:

$$\vec{F} = -k(r + d)\hat{r}$$

where $d$ is a constant. Find the modified expression for $U'(r)$ and show that the mass can execute a circular orbit with radius $r_0$, such that:

$$k(r_0 + d) = \frac{J^2}{mr_0^3}$$

(1.5 points)

f.) For the case $r_0 \gg d$, calculate the amount that orbits which are slightly perturbed from circular will precess. (2.5 points)

# Problem 2

A satellite is placed in a low-lying orbit using a two-stage rocket. The initial orbit is launched from the Earth's surface with $v_0 = 6\,\text{km/s}$ and angle $\theta_0 = 30\,\text{degrees}$. Upon reaching apogee, it is given a velocity boost to a new velocity $v_1$ that places the rocket into a circular orbit. (Use the numerical values $M_e = 6 \cdot 10^{24}$ kg, $R_e = 6371\,\text{km}$ in the following).

a.) Find the velocity at apogee before the velocity boost.

b.) Find the velocity at apogee after the velocity boost.

# Problem 3

A system has the Lagrangian

$$L = fy^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2 + y^2}$$

where $f$, $g$ and $k$ are constants. What is the Hamiltonian?