Electrodynamics

August 18, 2017

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

a.) State the Biot-Savart law and Ampere's theorem. For each, give an example of its use in the calculation of magnetic fields. (2 points)

b.) A circular loop of radius a carries a current I. Calculate the magnetic field on the axis of the loop a distance z from its centre. (2 points)

c.) By using $\vec{\nabla} \cdot \vec{B} = 0$, show that the radial component of the magnetic field at small distances ρ from the axis is:

$$B_\rho = \frac{3\mu_0 I a^2 z \rho}{4(z^2 + a^2)^{5/2}}$$

(3 points)

d.) An identical loop, also carrying current I in the same direction, is placed with its axis on that of the first loop, at a distance $d \gg a$. By considering the components of the magnetic field due to the first loop, calculate the magnitude of the force on the second loop. (2.5 points)

e.) In what direction does the force act? (0.5 points)

Hint: In cylindrical coordinates,

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

Problem 2

Specify *two* methods of measuring a static magnetic field $\vec{B}$ in vacuum or air, utilizing *different* physical principles in each of the methods. *Describe* the procedures involved in each method, using appropriate equations to show what quantities must be measured to determine $\vec{B}$.

Problem 3

Magnetic monopoles are introduced into electrodynamics by modifying one of Maxwell's equations to read $\vec{\nabla} \cdot \vec{B} = 4\pi\rho_M$. Set up the nonrelativistic equation of motion of an electrically charged particle of charge q about an isolated, fixed, pointlike magnetic monopole of strength Γ. Show that

$$\vec{Z} = \vec{r} \times \vec{p} - \frac{q\Gamma\vec{r}}{r}$$

is a constant of motion. Construct one further constant of motion.

Problem 4

a.) A grounded, conducting, infinite plane with a point charge q a distance d above it can be solved by the *method of images*. The induced charge density on the conductor is

$$\sigma(x,y) = \frac{-qd}{2\pi\left(x^2+y^2+d^2\right)^{3/2}}.$$

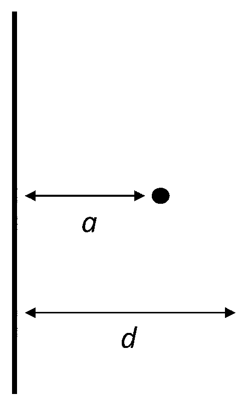
i.) Now consider a grounded, conducting, semi-infinite plane such that $0 \leq x < +\infty$ and $-\infty < y < +\infty$. If the point charge is placed at $(a, 0, d)$, then what is the total induced charge q_i on this plane?

You may need the following integrals,

$$\int \frac{dx}{\left(x^2+c^2\right)^{3/2}} = \frac{x}{c^2\sqrt{x^2+c^2}} \qquad\qquad \int \frac{dx}{\left(x^2+c^2\right)} = \frac{1}{c}\tan^{-1}\left(\frac{x}{c}\right)$$

ii.) How does this compare to the full infinite plane?

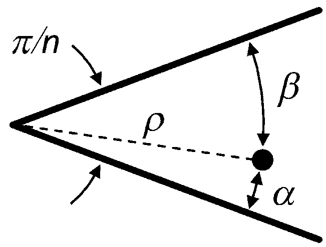
b.) Now consider the infinite, grounded, conducting parallel plates depicted in the following diagram. They are separated by distance d, and a point charge q is placed a distance a from the left plate. Place the origin on the left plate, at distance a from the point charge.



i.) Use the *method of images* to list all the image charges and their positions.

ii.) *Describe* how to determine the total induced charge *in principle*. You are *not* expected to fully evaluate the expression for the total induced charge.

c.) Now consider two infinite *grounded* plates that join at the origin as shown in the following diagram. They are separated by the angle π/n where n is an integer.



i.) Use the *method of images* to list all the image charges and their positions.

ii.) Discuss qualitatively the total induced charge as $n \to \infty$. You are *not* required to make an explicit calculation.

iii.) Discuss qualitatively how this problem relates to the parallel plate case in question b.).

Problem 5

A cyclotron is a device used to accelerate charged particles. It consists of two hollow, conducting drums (half cylinders, labeled D_1 and D_2 in the figure) where an alternating electric potential $V(t) = V_0 \sin(\omega t)$ is applied, together with a constant, uniform magnetic field $\vec{B}$ perpendicular to the flat surfaces of the cylinders. The spacing between the drums is $L \ll R$ so that the electric potential is approximately constant, equal to V_0, during the time a particle takes to traverse the gap; see figure for a drawing describing the configuration.

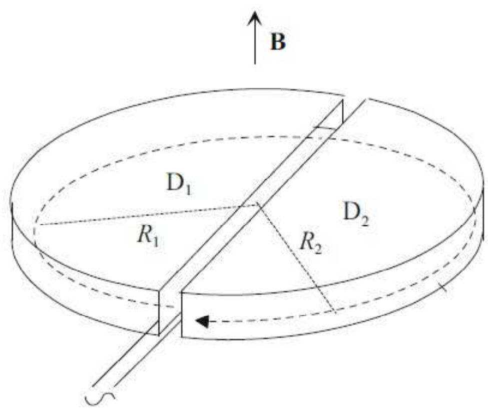


Figure 1: Principle of operation of the cyclotron

Each question below is worth 2 points.

a.) Particles are injected at the center with essentially zero velocity and are accelerated, moving in the electric and magnetic fields. When the polarity of the electric field is right, a particle will be pulled into one of the drums. Show that the particle will follow a semicircle inside one of the drums with constant radius R and derive the equation for R in terms of the particle mass m, charge q, and the velocity v, assuming non-relativistic kinematics. Justify any calculation by describing the electric field between and inside the drums, in words and with a rough sketch of the field lines.

b.) Calculate the kinetic energy gained by the particle at each full turn.

c.) As the velocity increases, the radius does as well, according to the equation derived in a.). Show that the time needed to complete a semicircle is constant and therefore a particle that starts with the electric field in the right phase will always be accelerated if the field frequency ω remains constant. Calculate the required frequency (the "cyclotron frequency") as a function of B, q, and m.

d.) When the radius of the particle trajectory becomes equal to the cyclotron radius R_0 the particle escapes. Calculate the kinetic energy of the extracted particle as a function of R_0, the field B, and the particle constants m and q.

e.) When the particle becomes relativistic, show that the cyclotron described above will no longer work and suggest one modification that will make it capable of continuing to accelerate relativistic particles.