Quantum Mechanics

August 17, 2017

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

The Hamiltonian for a two-state system is $H = H^0 + H^1$, where

$$H^0 = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix} \qquad\qquad H^1 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}.$$

Assume that b is real and positive; that b is small compared to E_1^0, E_2^0, and $E_2^0 - E_1^0$; and that $0 < E_1^0 < E_2^0$.

a.) Using perturbation theory, calculate the eigenvalues of H to order b^2 and the eigenvectors to order b.

b.) Using the variational method, estimate the lowest eigenvalue E_1 to order b^2 and the corresponding eigenvector to order b. Assume a trial function of the form $|\alpha\rangle = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$.

Problem 2

A Glauber coherent state is a special state of the simple harmonic oscillator (SHO). Coherent states are defined by the following eigenvalue/normalization relationship:

$$\begin{aligned} A|\alpha\rangle &= \alpha|\alpha\rangle \\ \langle\alpha|\alpha\rangle &= 1 \end{aligned}$$

where A is the SHO annihilation operator and α is not necessarily an integer state. Answer the following questions:

a.) Expand $|\alpha\rangle$ as a series expansion in $|n\rangle$.

b.) Calculate $\langle\alpha|\beta\rangle$.

As a reminder, SHO states obey:

$$\begin{aligned} A|n\rangle &= \sqrt{n}\,|n-1\rangle \\ A^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \langle n|n\rangle &= 1 \end{aligned}$$

Problem 3

Consider the (normalized) states, at $t = 0$, $|A, t=0\rangle = \alpha|0\rangle + \beta e^{i\phi}|1\rangle$ and $|B, t=0\rangle = \gamma|0\rangle + \delta e^{i\phi}|2\rangle$ of a one-dimensional, harmonic oscillator with angular frequency ω, where $|n\rangle = \frac{(-i)^n}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle$ are the eigenstates of the Hamiltonian and α, β, γ, and δ are real and non-zero (the arbitrary phase here has been chosen such that the corresponding wavefunctions are real).

a.) Show that the states $|A, t\rangle$ and $|B, t\rangle$ display a periodic evolution in time. Find the corresponding periods and compare with the classical period of the harmonic oscillator with this angular frequency.

b.) Calculate the time dependence of the expectation values of the following observables: position q; momentum p; total energy E; kinetic energy T; and potential energy V.

Problem 4

Devise a variational calculation of the polarizability of the hydrogen atom. A simple procedure would be to take as a trial function the linear combination $c_{1s}\psi_{1s} + c_{2pz}\psi_{2pz}$ (the basis could be lengthened in a more sophisticated treatment), or alternatively $(1+az)\psi_{1s}$, with a being the variation parameter. The Hamiltonian is $H = H_0 + ezE$ in each case. Find the best energy and identify α_{zz}. (Notation: The pz-orbital has magnetic quantum number $m = 0$).

Problem 5

a.) A particle of mass m is subject to a potential

$$V(\vec{r}) = -\frac{e^2}{r} + \lambda y$$

where e and λ are some parameters.

i.) identify a continuous spatial symmetry of the problem (1 point)

ii.) name the conserved quantity that corresponds to the symmetry identified in part i.) (0.5 points)

b.) For a one-dimensional harmonic oscillator, sketch the wave function $\psi(x)$ for the state with energy $E = \frac{5}{2}\hbar\omega$ (0.5 points)

c.) Is the following wave function

$$(x_1 - x_2)e^{-(\vec{r}_1 - \vec{r}_2)^2} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right)$$

allowed for two identical spin $\frac{1}{2}$ particles? Explain your answer! (1 point) d.) Positronium is a bound state of an e^+ and an e^-. What is the energy of the γ emitted in transitions of positronium from the first excited state to the ground state? Hint: the ground state energy of the hydrogen atom is approximately 13.6 eV. (1 point)

$E_\gamma =$

e.) A charged particle bound in a harmonic oscillator potential is placed into a uniform electric field. The energy of the 1^{st} excited state is (choose answer from list) (1 point)

i.) proportional to the electric field

ii.) quadratic in the electric field

iii.) some of the energy eigenvalues depend linearly on the electric field, and some quadratically

iiii.) independent of the electric field

f.) Consider a particle of mass m in a potential (in one dimension)

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & x > 0 \\ +\infty & x < 0 \end{cases}$$

What is the energy of the ground state? (1 point)

g.) In 3 dimensions, what is the ground state energy for a particle of mass m in an infinite spherical square well with radius R, i.e. $V(r) = \begin{cases} 0 & r < R \\ \infty & r > R \end{cases}$? Explain how you obtain your answer! (2 points) h.) What is the ground state energy for a system of two identical spin $\frac{1}{2}$ particles, each of mass m, both with spin $|\uparrow\rangle$ when a potential $V(\vec{r}_1 - \vec{r}_2) = \frac{k}{2}(\vec{r}_1 - \vec{r}_2)^2$ acts between them? (2 points)