Quantum Mechanics

August 16, 2018

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Electrons are placed into the ground state and first excited state of a simple harmonic oscillator (SHO). Calculate $\langle(\hat{x}_1 - \hat{x}_2)^2\rangle$ for the singlet and triplet spin states. In view of your results, how does accounting for the Coulomb repulsion influence the connection between spin and energy? Try to formulate a general qualitative statement. As a reminder, SHO states obey:

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$a = \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\hbar\omega}}\right)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\langle n|n\rangle = 1.$$

Problem 2

A non-relativistic particle of mass m moves in one dimension, subject to a potential energy function $V(x)$ which is the sum of three evenly spaced, attractive delta functions:

$$V(x) = -aV_0 \sum_{n=-1}^{1} \delta(x - na) \ , \qquad V_0, a > 0 \ .$$

a.) Calculate the discontinuity in the first derivative of the wavefunction at $x = -a, 0$, and a.

b.) In bound state wavefunctions for this system, how many nodes are possible in the region $x > a$?

c.) In bound state wavefunctions for this system, how many nodes are possible in the region $0 < x < a$?

d.) In bound state wavefunctions for this system, can there be a node at $x = a$?

e.) In bound state wavefunctions for this system, can there be a node at $x = 0$?

f.) If V_0 is large enough, the system will have both symmetric and antisymmetric bound states. Sketch qualitatively the lowest energy symmetric ($\psi(-x) = \psi(x)$) and antisymmetric ($\psi(-x) = -\psi(x)$) bound states.

g.) For V_0 much larger than other energy scales in the problem, how many bound states are there?

h.) For the lowest energy antisymmetric bound state, derive a transcendental equation that determines the bound state energy. You do not need to solve the equation.

Problem 3

An electron (mass m, charge $\hat{q} = -e$) is located within a homogeneous magnetic field $\vec{B} = B\vec{e}_z$, where B is constant and $\vec{e}_z$ a unit vector in the z-direction.

a.) (1 Point): What is the form of the classical Hamiltonian?

b.) (1 Point): Choose a vector potential $\vec{A}(\vec{x}) \equiv (0, A_y(x), 0)$ such that it satisfies the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$.

c.) (4 Points): Show that the eigenvalue problem of the Hamiltonian,

$$\hat{H}\psi = E\psi \ ,$$

can be solved by using the ansatz

$$\psi(\vec{r}) = \psi(x, y, z) = e^{ik_z z} e^{ik_y y} \varphi(x) \ ,$$

and that it can be reduced to the eigenvalue problem of the linear harmonic oscillator.

d.) (2 Points): Determine the eigenenergies and the eigenfunctions.
Note: The eigenenergies and eigenfunctions of the linear harmonic oscillator are well-known, you do **not** actually need to solve the differential equation. Simply apply the known results.

e.) Now assume that – in addition to the magnetic field – a force $\vec{F} = -m\omega^2 z\vec{e}_z$ acts on the electron in z-direction. Repeat all of the steps above for this situation:

i.) (1 Point): Find the Hamiltonian and choose a vector potential as in b.) which satisfies the Coulomb gauge.

ii.) (2 Points): Find a suitable ansatz for the eigenfunction $\psi(\vec{r}) = \psi(x, y, z)$ of the Hamiltonian.

iii.) (2 Points): Find the explicit form of the eigenvalues and eigenfunctions.

Problem 4

a.) For two identical spin-$\frac{1}{2}$ particles in one dimension, is the following an allowed wave function,

$$\psi_{s_1 s_2}(x_1, x_2) = \mathcal{N}(x_1 - x_2)e^{-(x_1-x_2)^2} \cdot (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

($\mathcal{N}$ is an appropriate normalization factor)? You must explain how you reach your conclusion! (1 point)

b.) An electron is in a state described by the wave function (in spherical coordinates)

$$\psi(r, \theta, \phi) = \mathcal{N} r e^{-ar} \sin(\theta)\cos(\phi) \ ,$$

where $\mathcal{N}$ is a normalization factor such that the wave function is normalized to 1. Suppose L_z is measured in this state. What are the possible outcomes for the measurement of L_z (list only those with nonzero probability) and what are the respective probabilities? (1 point)

The following spherical harmonics may be useful when solving this problem:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos(\theta) \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2(\theta)e^{\pm 2i\phi}$$

$$Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin(\theta)\cos(\theta)e^{\pm i\phi} \qquad Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2(\theta) - 1\right)$$

c.) Consider the ground state of the harmonic oscillator in 3 dimensions with an external electric field applied, $\Delta V = -eEz$, as a perturbation. In 1^{st} order perturbation theory, the energy of the ground state (1 point)

i.) increases

ii.) decreases

iii.) stays the same

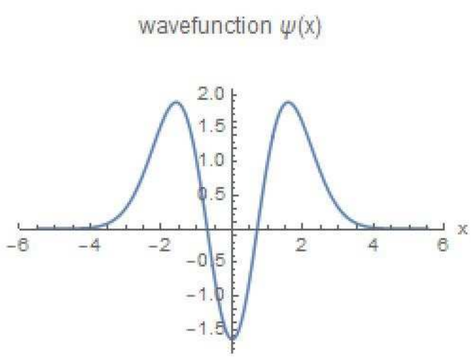
In 2^{nd} order perturbation theory, the energy of the ground state (1 point)

i.) increases

ii.) decreases

iii.) stays the same

d.) Shown is one of the wave functions for a particle of mass m in one dimension subject to a harmonic oscillator potential $V(x) = \frac{k}{2}x^2$. What is the energy of that state? Explain your answer! (1 point)



e.) In 3 dimensions, what is the ground state energy for a particle of mass m in an infinite square well with radius R, i.e. (1 point)

$$V(\vec{r}) = \begin{cases} 0 & |\vec{r}| < R \\ +\infty & |\vec{r}| > R \end{cases}$$

Explain your answer!

f.) Two identical spin-$\frac{1}{2}$ particles of mass m in 3 dimensions, both with spin $\uparrow$, interact with a potential energy

$$V(\vec{r}_1, \vec{r}_2) = \frac{k}{2}(\vec{r}_1 - \vec{r}_2)^2 \ ,$$

where k is a constant. What is the energy of the ground state of that system? Explain your answer! (4 points)

Problem 5

Evaluate the differential scattering cross section in a repulsive field,

$$V = \frac{A}{r^2}$$

in the Born approximation. Discuss under what conditions your result is applicable.