Classical Mechanics

August 24, 2019

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

A bead of mass m is constrained to move on a frictionless wire in the shape of a cycloid described by the parametric equations

$$x = a(\phi - \sin\phi) \qquad \text{and} \qquad y = a(1 - \cos\phi)$$

subject to gravity, where x is a horizontal and y the vertical coordinate; a denotes a constant. Consider specifically the case that the bead starts from rest at height $y = 2a$.

a.) Find the speed of the bead at the bottom, $y = 0$.

b.) Find the period of oscillation.

Problem 2

Prove that if a particle moves in a central force field, then its path must be a plane curve.

Problem 3

A bead of mass m slides freely on a light wire of parabolic shape, which is forced to rotate with angular velocity ω about a vertical axis. The system is subject to gravity. The parabola is described by

$$z = \frac{1}{2}ar^2$$

where z is the height and r is the distance from the axis of rotation; a denotes a constant.

a.) Show that the Lagrangian for this system is:

$$L = \frac{1}{2}m\left[(1 + a^2r^2)\dot{r}^2 + (\omega^2 - ag)r^2\right]$$

b.) Find a constant of the motion.

c.) The bead is released at $r = 1/a$ with given $\dot{r} = v$. Show that, if $\omega^2 \geq ag$, the bead escapes to infinity. Show that, if $\omega^2 < ag$, it oscillates about $r = 0$, and find the maximum value of r.

d.) The wire now instead is allowed to rotate freely about the vertical axis with angular velocity $\dot{\phi}$. Find the new Lagrangian and constants of the motion.

e.) If the bead is released with the original initial conditions, show that it cannot escape to infinity for any value of ω. Find the maximum and minimum values of r.