Electrodynamics

August 23, 2019

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

The complex impedance Z of an electromagnetic wave propagating along the unit vector $\hat{n}$ in a medium is defined by the relationship

$$\vec{H} = \hat{n} \times \frac{\vec{E}}{Z}$$

between the magnetic and electric field strengths $\vec{H}$ and $\vec{E}$.

Starting with Maxwell's equations for inhomogeneous plane waves, show that the impedance Z for an isotropic homogeneous medium with non-zero complex dielectric constant ϵ and permittivity $\mu = 1$ is given by

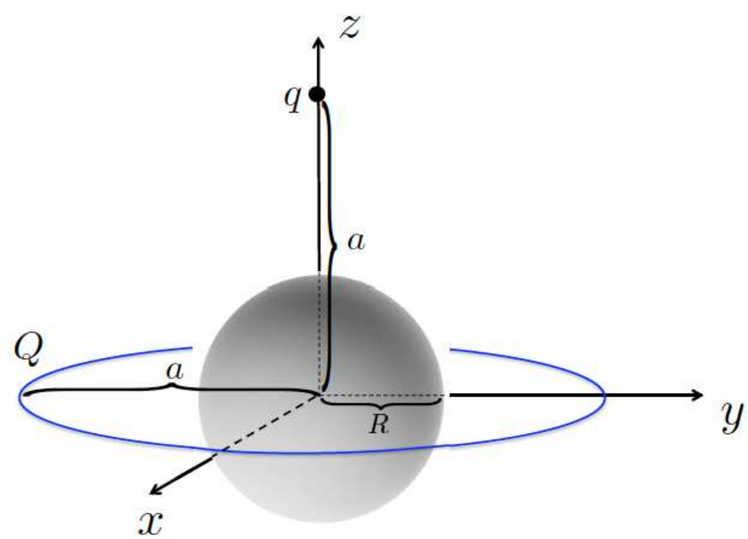
$$Z = \frac{Z_0}{n},$$

where $n = \sqrt{\epsilon}$ is the complex refractive index and Z_0 the impedance of the vacuum.

Number important equations and explain critical steps in your solution.

Hint: What are the current and the charge density in the medium through which the wave propagates?

Problem 2



The Figure shows a conductive ball of radius R. The electrostatic potential on the surface of the ball may vanish. In addition, a homogeneously charged ring is located in the (x, y) plane with radius $a > R$. The origins of ball and ring coincide. Assume that the ring is so thin that its charge may be considered a line charge. Also, a point charge q is located on the z-axis at the same distance a.

a.) **(2P)** Determine the charge density $\rho(\vec{x})$ outside of the ball.

b.) **(4P)** If you want to solve the electrostatic problem by means of the method of *image / mirror charges*, what image charges are needed? What is the total charge on the ball?
Hint: It might be helpful to consider the ring as a "chain" of small point charges.

c.) **(4P)** How would you choose the charge Q so that no force in z-direction acts on the point charge q? In this case, does a force act at all on the point charge?
Reminder: The Green's function that respects the Dirichlet boundary condition for the geometry that we are dealing with in this problem reads (with $r' = |\vec{x}'|$)

$$G_D(\vec{x}, \vec{x}') = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{|\vec{x} - \vec{x}'|} - \frac{R/r'}{|\vec{x} - (R^2/r'^2)\, \vec{x}'|} \right).$$

Problem 3

The action for a system consisting of a relativistic charged particle moving in an electromagnetic field is given by

$$S = -\int mc^2 d\tau - \int eA_\mu dx^\mu(t)$$

where $x^\mu = (ct, \vec{x})$, $A_\mu = (\phi/c, \vec{A})$, and τ is the proper time.

a.) Derive the equations of motion in terms of the electric and magnetic fields,

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} \qquad \text{and} \qquad \vec{B} = -\vec{\nabla} \times \vec{A}$$

b.) For $\vec{B} = 0$ and constant $\vec{E}$, at time $t = 0$ the particle has velocity $\vec{v}_0$. Find the subsequent velocity of the particle.

c.) Find the limiting velocity of the particle as $t \to \infty$.

d.) For $\vec{E} = 0$, show that γ, and hence the total speed, are constant.

e.) For $\vec{E} = 0$ and constant $\vec{B}$, show that the time dependence of the perpendicular velocity vector

$$\vec{v}_\perp = \vec{v} - \vec{B}\,\frac{\vec{v} \cdot \vec{B}}{B^2}$$

is periodic and find the period.

Problem 4

Starting from the vector potential of a pure magnetic dipole of dipole moment $\vec{m}$ positioned at the origin,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

derive the complete corresponding magnetic field $\vec{B}(\vec{r})$. Take special care concerning the singularity at the origin, by considering the integral of the magnetic field *vector* over a whole solid sphere centered on the origin and employing a variant of Gauß' theorem applicable for curls. Derive the latter by inserting the form $\vec{v}(\vec{r}) \times \vec{c}$ for the vector field in Gauß' theorem, with $\vec{c}$ an *arbitrary* (!) constant vector. Note: The contribution you are deriving here enters the hyperfine splitting in atomic spectra.

Problem 5

There is a layer of ionized gas surrounding the Earth, called the ionosphere, starting at an altitude $H \approx 300$ km above the surface. It has a wavelength-dependent refractive index that can be written approximately as

$$n_i = \sqrt{1 - \frac{\lambda^2}{C}},$$

where $C \approx 10^3$ km^2 is a constant. The Earth's radius is $R \approx 6370$ km. Calculate

1. The smallest radio frequency that can be used for communications with a satellite, and
2. The largest frequency that can be used for station-to-station communications using the ionosphere,

assuming that it has a sharp boundary at the altitude mentioned above.