Quantum Mechanics

August 22, 2019

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Two electrons are tightly bound to different neighboring sites in a certain solid. They are therefore distinguishable particles, which can be described by their respective Pauli spin matrices $\sigma^{(1)}$ and $\sigma^{(2)}$. The Hamiltonian of these two electrons takes the form

$$H = -J\left(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)}\right)$$

where J is a constant.

a.) How many energy levels does the system have? What are their energies? What is the degeneracy of the different levels?

b.) Now add a magnetic field in the z-direction. What are the new energy levels? Draw an energy level diagram as a function of B_z.

Problem 2

A particle of mass M and energy $E > 0$ scatters off a three-dimensional potential of radius R, i.e.,

$$V(r) = \begin{cases} V_0 & , r \leq R \\ 0 & , r > R \end{cases} , \; V_0 > 0 \, .$$

a.) **(2P)** Derive the radial equation for $u_l(r)$ from the stationary Schrdinger equation, $\hat{H}\varphi_{lm} = E\,\varphi_{lm}$, and the partial wave ansatz $\varphi_{lm}(\vec{x}) = \frac{u_l(r)}{r} Y_{lm}(\Omega)$.

b.) **(4P)** Derive the following relation for the s-wave ($l = 0$) and for energies $E < V_0$:

$$k \, \cot(kR + \delta_0) = \eta \, \coth(\eta R) \, ,$$

where $\eta = \sqrt{K_0^2 - k^2}$, $K_0^2 = \frac{2mV_0}{\hbar^2}$, $k^2 = \frac{2mE}{\hbar^2}$.

c.) **(4P)** Find expressions for the phase shift $\delta_0(E)$, the scattering length a and the total cross section $\sigma_0(E)$ in the limit of small energies. Again, assume s-wave scattering.

Hints:

$$\Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{1}{r^2}\frac{\vec{L}^2}{\hbar^2} \, , \quad a = -\lim_{k \to 0} \frac{\delta_0(k)}{k} \, , \quad \mathrm{arccot}(\tfrac{1}{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \, .$$

Problem 3

Consider a particle of mass m confined to a Möbius strip, i.e., a rectangular area of length L and width W with appropriate boundary conditions: Using the coordinate x to label position along the width of the strip, $x \in [-W/2, W/2]$, the particle encounters a hard boundary at the open edge, i.e., at $x = \pm W/2$; on the other hand, in the coordinate y labeling position along the length, $y \in [0, L]$, the wave function experiences the Möbius twist, $\psi(x, L) = \psi(-x, 0)$. Apart from being confined to the Möbius strip, the particle is not influenced by any potentials.

Determine the complete spectrum of stationary wave functions and associated energies of the particle.

Note: Treat this as a purely two-dimensional problem, as formally described above, i.e., do not imagine the Möbius strip to be embedded in three-dimensional space, with the particle carrying out a three-dimensional motion constrained to the two-dimensional Möbius surface (cf. the more familiar case of periodic boundary conditions in both directions, which is often referred to as a torus, even as it is likewise not necessarily treated as being embedded in three-dimensional space).

Problem 4

A biradical is prepared with two electrons in a singlet spin state. A magnetic field is present, and because the two electrons are in different environments, their interaction with the field is

$$H_{\text{int}} = \frac{\mu_B}{\hbar} B \left(g_1 s_{z1} + g_2 s_{z2} \right)$$

with $g_1 \neq g_2$. Evaluate the time dependence of the probability that the electron spins will take on a triplet configuration (i.e., that the $S = 1, m_S = 0$ state will be populated). Examine the role of the energy separation $\hbar J$ of the singlet state and the $m_S = 0$ state of the triplet. Suppose $g_1 - g_2 \approx 10^{-3}$ and $J = 0$; how long does it take for the triplet state to emerge in a field of $B = 1\,T$?

Hint: The probability of finding the system in one of the states as a function of time takes the Rabi form,

$$P_2 = |a_2|^2 = \frac{4V^2}{\omega_0^2 + 4V^2} \sin^2 \frac{1}{2} \sqrt{\omega_0^2 + 4V^2}\, t$$

where ω_0 and V are determined by the parameters of the interaction Hamiltonian. The coupled spin states are

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \right) \qquad |1,0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle \right)$$

Problem 5

a.) A particle of mass m is subject to a potential (in three dimensions)

$$V(\vec{r}) = -\frac{1}{x^2 + z^2}$$

List all quantities (or their corresponding operators) that are conserved. (1 point)

b.) A deuterium atom (D) is like the hydrogen atom (H), except that the nucleus consists of a proton and a neutron, i.e., D has about twice the mass of H. Which of the following is true for the ground state binding energy for the e^- in D: The energy required to excite D from the ground state to the 1^{st} excited state is: (1 point)

i.) about half that in H

ii.) slightly smaller than in H

iii.) the same as in H

iv.) slightly larger than in H

v.) about twice that in H

Explain your answer!

c.) Consider the ground state of an infinite square well in one dimension,

$$V(x) = \begin{cases} 0 & |x| < \frac{L}{2} \\ \infty & |x| > \frac{L}{2} \end{cases}$$

i.) Suppose the particle has mass m, what is the energy of the ground state? (1 point)

ii.) Now an external electric field $\Delta V = -eEx$ is applied as a perturbation. In 1^{st} order perturbation theory, the energy of the ground state: (1 point)

α.) increases

β.) decreases

γ.) stays the same

iii.) Again an external electric field $\Delta V = -eEx$ is applied as a perturbation. In 2^{nd} order perturbation theory, the energy of the ground state: (1 point)

α.) increases

β.) decreases

γ.) stays the same

d.) Consider now instead two particles, both of mass m, in one dimension bound to each other by a potential

$$V(x_1, x_2) = \begin{cases} 0 & |x_1 - x_2| < \frac{L}{2} \\ \infty & |x_1 - x_2| > \frac{L}{2} \end{cases}$$

i.) What is the energy of the ground state if the two particles are distinguishable? (1 point)

ii.) What is the energy of the ground state if the two particles are indistinguishable spin $\frac{1}{2}$ particles, with both spins parallel? (1 point)

e.) Consider now a particle of mass m in three dimensions confined inside a sphere of radius R, i.e.

$$V(\vec{r}) = \begin{cases} 0 & |\vec{r}| < R \\ \infty & |\vec{r}| > R \end{cases}$$

What is the energy of the ground state? (3 points)