Quantum Mechanics

August 20, 2020

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

In the following problem we consider a 2-fermion system with wave function $\psi(1,2)$.

a.) What condition must this wave function satisfy if the two particles are identical? Explain.

b.) How does the statement in a.) imply the statement that no two electrons in an atom can have identical quantum numbers? Explain.

c.) The first excited state of Mg (Magnesium) has a valence electron configuration with one electron in a $3s$ orbital state and one in a $3p$ orbital state $(3s, 3p)$. In the strong spin-orbit coupling limit (LS-limit), which values of the quantum numbers l and s for the system are possible? What is the form of the spatial part of the corresponding wave functions in terms of the single-particle wave functions $\psi_s(\vec{r})$ and $\psi_p(\vec{r})$? Explain which of those states will have the lowest energy.

Problem 2

An atom has an unfilled d-shell with total spin $S = 1$ and orbital angular momentum $L = 1$. The energy eigenfunctions are characterized by the total angular momentum $\vec{J} = \vec{L} + \vec{S}$.

a.) In terms of the eigenfunctions $|L, L_z\rangle$ of orbital angular momentum, and the eigenfunctions $|S, S_z\rangle$ of spin, write down the energy eigenfunctions of the five states with $J = 2$, the three states with $J = 1$, and the single state with $J = 0$.

b.) Suppose this term exists in the Hamiltonian:

$$H = A\vec{J} \cdot \vec{J} + B\vec{L} \cdot \vec{S} \; .$$

Show that the $J = 0$ state is lowest in energy, with the degenerate $J = 1$ triplet at an energy Δ above the singlet, and the 5-fold degenerate $J = 2$ states at an energy Δ' above the singlet. Calculate the values Δ and Δ' in terms of A and B.

c.) Now consider the atom in the presence of a small magnetic field B_0, which contributes to the energy of the atom via this magnetic moment term:

$$H' = -\frac{e\hbar}{2mc}\left(\vec{L} + 2\vec{S}\right) \cdot \vec{B}_0 = -\vec{\mu} \cdot \vec{B}_0 \; .$$

Find the expectation value of the magnetic moment $\vec{\mu}$ in the ground state correct to lowest order in B_0.

Problem 3

Calculate the expectation value of the angular momentum of a particle moving in two dimensions trapped in an infinite square well:

$$V = \begin{cases} 0 & -a < x < a,\ -a < y < a \\ \infty & \text{everywhere else} \end{cases}$$

for

a.) the ground state and

b.) the first excited state

Problem 4

The rotation of the HI molecule can be pictured as an orbiting of the Hydrogen at a radius of 160 pm about a virtually stationary Iodine atom. If the rotation is thought of as taking place in a plane, what are the rotational energy levels? What is the wavelength of the radiation emitted in the transition $m_l = 1 \longrightarrow m_l = 0$?

Problem 5

a.) A deuterium atom (D) is like the hydrogen atom (H), except that the nucleus consists of a proton and a neutron. Which of the following is true for the ground state binding energy for the e^- in D: The energy required to excite D from the ground state to the 1^{st} excited state is (2 points)

i.) about half that in H

ii.) slightly smaller than in H

iii.) the same as in H

iv.) slightly bigger than in H

v.) about twice that in H

Explain your answer:

b.) The energy spectrum of ordinary hydrogen is given by $E_n = -13.6\,\text{eV}\frac{1}{n^2}$. What would the energy spectrum be for muonic hydrogen, where the e^- is replaced by a μ^- with a mass of 207 times the e^- mass? (2 points)

Explain your answer! $m_e c^2 = 0.511\,\text{MeV}$, $m_\mu c^2 = 106\,\text{MeV}$, $m_p c^2 = 938\,\text{MeV}$

c.) Consider two spin $\frac{1}{2}$ particles bound together by a harmonic oscillator potential in three dimensions, i.e.

$$V(\vec{r}_1, \vec{r}_2) = \frac{1}{2} k \left(\vec{r}_1 - \vec{r}_2\right)^2$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the two particles. The spin part of the wave function reads $\frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$. What is the energy spectrum for that system? (3 points)

Note: if you write down an equation for E_n, be specific about the values of n.

d.) Consider a harmonic oscillator in two dimensions, $V(x, y) = \frac{k}{2}\left(x^2 + y^2\right)$.

i.) What is the energy of the first excited state? (1 point)

ii.) Now a perturbation $\Delta V(x, y) = \lambda \delta(x - a)\delta(y - a)$ is applied, where a is some constant. Using degenerate perturbation theory, what would be the change in energy for the first excited state? (2 points)

Hint: the unperturbed wave functions for the ground state and the 1^{st} excited state of a one-dimensional harmonic oscillator in one dimension read respectively $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$, and $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2}$