Electrodynamics

August 20, 2021

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Consider a point charge q traveling at a velocity $\vec{v}$ directly away from some point P.

a.) Determine the electric field and magnetic field at the point P, which is some instantaneous distance $\vec{r} = \vec{v}t = vt\hat{r}$ away from the point charge. Assume the velocity is non-relativistic for this part.

b.) Now assume that the charge is moving at a relativistic speed. Calculate the new electric field by boosting to a frame where β is significant. Assume the initial velocity is negligible compared to the new relativistic velocity, and the direction of the velocity is the same. *Hint:* You will need the boost transformation $\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta} \cdot \vec{E})\vec{\beta}$, and to transform $t' = \gamma t$ to account for Lorentz dilation.

c.) Show that as $\gamma \to 1$ or $\beta \to 0$, the original electric field is recovered. Discuss briefly how the relativistic field differs from the non-relativistic field at the point P for a charge moving in-line. Qualitatively, how do you expect the field might change for a point displaced perpendicularly from the path of the charge, as the charge passes by?

Problem 2

Consider a plane electromagnetic wave of frequency ω propagating through a non-magnetic conducting medium of conductivity σ. Assume that the wave is traveling in the z direction, the initial amplitude of the wave $E|_{z=0} = E_0$, and the medium is a good conductor, i.e. $\sigma \gg \epsilon_b \omega$.

a.) Find the electric and magnetic fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ inside the conductor.

b.) Calculate the time-averaged Poynting vector $\mathbf{S}(z)$.

c.) Show that the power per unit volume lost to Joule heating in the conductor equals $-\partial S/\partial z$.

Problem 3

Two charges with the same magnitude but opposite signs, $+q$ and $-q$, are placed on opposite sides of a circle with radius a; thus, they are separated by a distance $d = 2a$.

Now, assume that the circle is in the $x - y$ plane and the two charges orbit around the z-axis through the center of the circle with a common frequency ω.

a.) Express the electric dipole moment $\vec{p}$ of such a system with two circulating opposite charges as a function of time using Cartesian coordinates.

b.) With $d \ll c/\omega$ and for points far away from the moving charges, i.e., $r \gg d$, the average radiated power can be computed for the general case via the average Poynting vector $\vec{S}$ and one finds

$$\langle dP \rangle = \langle \vec{S} \rangle \cdot d\vec{A} = \frac{\mu_0}{16\pi^2 c} \langle (\ddot{\vec{p}} \times \hat{r})^2 \rangle \; d\Omega$$

Use your results from part a.) to compute the angular distribution of the radiated power for the above system of the two circulating charges.

c.) How does this result compare to the power radiated by a simple oscillating electric dipole?

Problem 4

Consider a pointlike electric charge e and a pointlike magnetic monopole g placed at a fixed distance a from each other.

a.) Write down the electric and magnetic fields.

b.) Write down the momentum and angular momentum densities of the electromagnetic field.

c.) What is the direction of the total (integrated) angular momentum $\vec{L}$?

d.) Does $\vec{L}$ depend on a?

e.) Calculate the magnitude of $\vec{L}$. *Hint: For any vector* $\vec{B}$, $\frac{1}{r}(\vec{B}-\hat{r}(\vec{B}\cdot\hat{r})) = (\vec{B}\cdot\vec{\nabla})\hat{r}$.

Problem 5

Consider an empty spherical cavity of radius R inside an infinite conducting volume as shown in Fig. 1. The conductor is grounded such that the electrostatic potential $\Phi = 0$ vanishes throughout. A point charge q is placed inside the cavity at a displacement $z\hat{k}$ from the center along the z-axis.

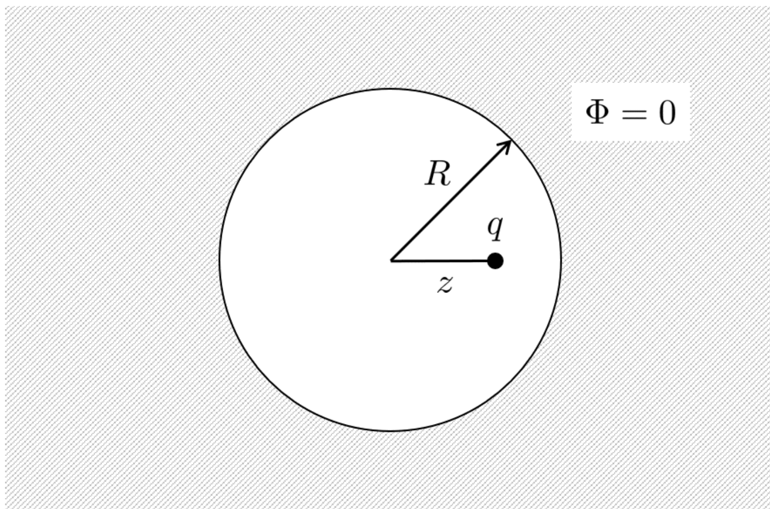

Figure 1: A spherical cavity inside an infinite, grounded conducting volume.

a.) *(1 pt.)* What is the method of images, and why does it work? Explain briefly in words.

b.) *(2 pts.)* Find the charge q' and position z' of an image charge (also along the z-axis) which satisfies the necessary boundary conditions.

c.) *(1 pt.)* What is the resulting electrostatic potential inside the cavity?

d.) *(2 pts.)* What is the resulting electric field inside the cavity? *(Note: The expression in polar coordinates is straightforward but messy; you don't need to simplify it.)*

e.) *(1 pt.)* What is the resulting electric field *outside* the cavity? *(Hint: Trick question!)*

f.) *(2 pts.)* Show that the charge density σ induced on the surface of the conducting volume is

$$\sigma = \frac{-q}{4\pi R^2} \frac{1 - \frac{z^2}{R^2}}{\left(1 + \frac{z^2}{R^2} - 2\frac{z}{R}\cos\theta\right)^{3/2}}, \tag{1}$$

where θ is the polar angle in spherical coordinates.

g.) *(1 pt.)* *Without calculating anything*, what is the net charge induced on the surface of the conducting volume? Why?

Note: the gradient in spherical coordinates is given by

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}$$