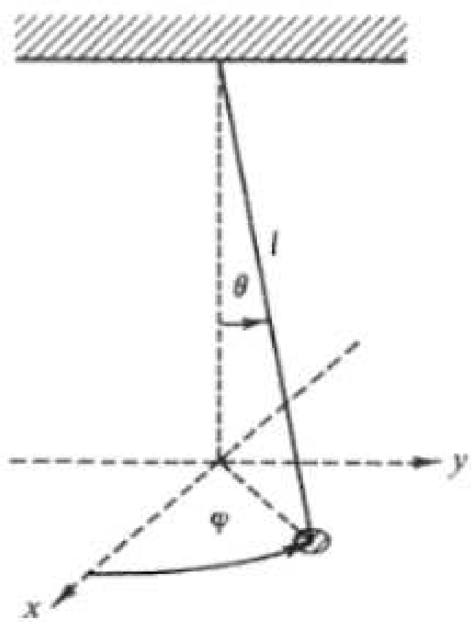
Classical Mechanics

August 20, 2022

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

A mass m is suspended from a wire of length l and it's free to move in polar angle θ and azimuthal angle ϕ. The system is subject to gravity.



a.) Write the Lagrangian for this system.

b.) Derive the Lagrange equations of motion.

c.) One of the equations found in b.) is an expression of a conservation principle. Explain.

Problem 2

Two point masses m are each constrained to slide (without friction) on the same fixed wire of circular shape lying in a vertical plane. The radius of the circle is R. In addition, the two masses are rigidly connected by a massless rod of length l (of course, $l \leq 2R$). The system is subject to gravity.

a.) Construct a Lagrangean for the system and derive the equation of motion (do not attempt to solve it in full generality).

b.) Give the angular frequency ω of small oscillations about the stable equilibrium position (if there is one). Discuss ω as a function of l for fixed R, and also as a function of R at fixed l. Interpret the relevant limits. Is there a maximal ω in the latter case, for fixed l?

Problem 3

Aspects of the Central Force Problem: Consider a system of two particles with masses m_1 and m_2 which interact according to a potential

$$V(\vec{r}_1\,,\,\vec{r}_2) = \frac{-k}{|\vec{r}_1 - \vec{r}_2|^\alpha}\,, \tag{1}$$

where k and α are constants. For the special case $\alpha = 1$ this corresponds to the familiar $1/r$ dependence describing the Coulomb or gravitational potential.

a.) *(1 pt.)* Using the positions $\vec{r}_1, \vec{r}_2$ of the individual masses m_1, m_2 respectively as your independent variables, find the *Lagrangian* and the *equations of motion* for this system.

b.) *(2 pts.)* Perform the change of variables to the center-of-mass position $\vec{R}$ and relative displacement $\vec{r}$, defined by

$$\vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\,, \qquad \vec{r} \equiv \vec{r}_1 - \vec{r}_2\,. \tag{2}$$

Find the *Lagrangian* and the *equations of motion* in terms of these new coordinates.

c.) *(1 pt.)* Show that $\vec{R}$ is a *cyclic coordinate* and identify the associated *constant of the motion.*

d.) *(2 pts.)* Working in the center-of-mass reference frame (such that the center of mass $\vec{R} = 0$ is at rest at the origin), change variables from the Cartesian components (r_x, r_y, r_z) of the displacement vector to the spherical coordinates (r, θ, ϕ). Find the *Lagrangian* in terms of these new coordinates. You may find it useful to have the spherical-coordinate basis vectors:

$$\hat{e}_r \equiv \frac{\vec{r}}{r} = \sin\theta\,\cos\phi\,\hat{i} + \sin\theta\,\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\,, \tag{3}$$

$$\hat{e}_\theta = \cos\theta\,\cos\phi\,\hat{i} + \cos\theta\,\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\,, \tag{4}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\,. \tag{5}$$

e.) *(1 pt.)* Show that ϕ is a *cyclic coordinate* and identify the associated *constant of the motion*. What does this imply for the *other components* of this vector quantity?

f.) *(3 pts.)* Use the equations of motion (in whatever choice of variables you like) to show that there is an additional constant of motion – the *Laplace-Runge-Lenz vector*

$$\vec{A} \equiv \vec{p} \times \vec{L} - \mu k \, \hat{e}_r \, . \tag{6}$$

Show that $d\vec{A}/dt = 0$, but *only for the Coulomb potential* $\alpha = 1$.