Electrodynamics

August 19, 2022

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

A slowly varying magnetic field, $B = B_0 \cos(\omega t)$, pointing into the y-direction, induces eddy currents in a slab of material occupying the half-space $z > 0$. The slab has permeability μ and conductivity σ. Starting from Maxwell's equations, determine the attenuation of the eddy currents with depth into the slab and the phase relation between the eddy currents and the inducing magnetic field.

Problem 2

A thin disk of radius R is centered at the origin and lies in the $x - y$ plane. The disk carries a uniform distribution of electric dipole moment density $\mathbf{p} = p\hat{\mathbf{z}}$ per unit area. Calculate the electrostatic potential $\Phi(z)$ along the axis of the disk.

Problem 3

Consider the transformation of electromagnetic fields under a boost to a new frame with constant relative speed β.

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta} \cdot \vec{E}) \tag{1}$$

$$\vec{B}' = \gamma(\vec{B} + \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta} \cdot \vec{B}) \tag{2}$$

a.) If there exists an initial non-zero electric field but no initial magnetic field, does there exist a frame in which there is a zero electric field but a non-zero magnetic field? If so, find the angle between $\vec{\beta}$ and $\vec{E}$ that produces such a frame.

b.) Show that, for a non-zero electric field and zero magnetic field, to boost to a frame with arbitrary non-zero electric and magnetic field, the boost angle between $\vec{\beta}$ and $\vec{E}$ is independent of the initial E field amplitude and only dependent on the ratio of the boosted field amplitudes E'/B'.

c.) Show that charge is conserved under frame transformation to a new frame with constant relative speed β. *Hint:* keep in mind that this is a relativistic transformation.

d.) We can similarly move to a rotating frame, under which a particle will experience a torque $\frac{d\vec{s}}{dt} = \vec{\mu} \times \vec{B}'$ where $\vec{s}$ is the spin and $\vec{\mu} = \mu\vec{s}$ is the magnetic moment. The energy of the transformed system is then $U' = \vec{\mu} \cdot \vec{B}'$. Let's consider a radially symmetric electric field and a non-relativistic boost speed. Show under these conditions that the energy after transformation has two terms: one dependent on $\vec{s} \cdot \vec{B}$ and another on $\vec{s} \cdot \vec{L}$, where $\vec{L}$ is the angular momentum.

Problem 4

Consider a point charge e that moves along the z-axis with speed v in a laboratory with an observer located at $(x_0, 0, 0)$. The clocks in the laboratory frame and in the rest frame of the charge show zero at the point of closest approach.

a.) By starting in the rest frame of the charge and transforming appropriately, find the scalar and vector potentials ϕ and $\vec{A}$ at the position of the observer in the laboratory. (Note: For the purposes of b.) below, it may be useful in an intermediate step to keep the position at which the fields are evaluated more general than the specific location of the observer.)

b.) Find also the electric and magnetic fields at the position of the observer in the laboratory.

Problem 5

Electromagnetic Waves in Medium:

a.) *(1 pt.)* Consider a uniform, isotropic, linear, nondispersive medium – that is, one for which the electric displacement $\vec{D}$ and magnetic field $\vec{H}$ are simply related to the electric field $\vec{E}$ and magnetic induction $\vec{B}$ by

$$\vec{D} = \epsilon \vec{E} \qquad\qquad \vec{B} = \mu \vec{H} \tag{1}$$

with the electric permittivity ϵ and magnetic permeability μ constants. Express *Maxwell's equations* in terms of $\vec{E}$ and $\vec{B}$ *in this medium* (in the absence of any explicit charges or currents).

b.) *(2 pts.)* Solve Maxwell's equations in the medium (1) by inserting the Fourier transform

$$\begin{aligned}
\vec{E}(\vec{r}, t) &\equiv \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3}\, e^{-i\omega t}\, e^{+i\vec{k}\cdot\vec{r}}\, \vec{E}(\vec{k}, \omega)\,, \\
\vec{B}(\vec{r}, t) &\equiv \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3}\, e^{-i\omega t}\, e^{+i\vec{k}\cdot\vec{r}}\, \vec{B}(\vec{k}, \omega)\,.
\end{aligned} \tag{2}$$

Show that *any Fourier mode* $\vec{E}(\vec{k}, \omega)$ is a *solution of Maxwell's equations*, so long as

$$k = \sqrt{\mu\epsilon}\, \omega\,. \tag{3}$$

This shows that the solutions of Maxwell's equations in this medium are plane waves. What is the *speed* of this plane wave?

c.) *(1 pt.)* Suppose an electromagnetic plane wave that is propagating through the medium (1) has an electric field of the form

$$\vec{E}(\vec{r}, t) = \vec{E}_0\, e^{-i\omega t}\, e^{+i\vec{k}\cdot\vec{r}} \tag{4}$$

(that is, a single Fourier mode). Show that Maxwell's equations in the medium imply that the magnetic field is propagating *in phase* with the electric field:

$$\vec{B}(\vec{r}, t) = \vec{B}_0\, e^{-i\omega t}\, e^{+i\vec{k}\cdot\vec{r}}\,, \tag{5}$$

and *express* $\vec{B}_0$ *in terms of* $\vec{E}_0$.

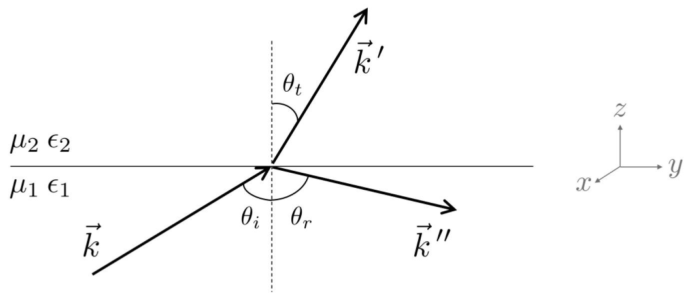

Figure 1: An electromagnetic wave with wave vector $\vec{k}$ is incident on an interface between two dielectric media as shown, producing both a reflected wave (wave vector $\vec{k}''$) and a transmitted wave (wave vector $\vec{k}'$). The interface lies in the xy-plane.

d.) *(1.5 pts.)* Suppose now that an electromagnetic plane wave with wave vector $\vec{k}$ is incident on a boundary between two linear media of the type (1), as shown in Fig. 1. The boundary is the plane $z = 0$, with the medium having constants ϵ_1, μ_1 for $z < 0$ and ϵ_2, μ_2 for $z > 0$. Maxwell's equations imply certain *boundary conditions* on the normal and in-plane components of the electric and magnetic fields. Without calculating anything, of the four different components

$$\text{Normal}: \quad E_z \quad , \quad B_z \quad , \tag{6}$$

$$\text{In} - \text{Plane}: \quad \vec{E}_\perp \quad , \quad \vec{B}_\perp \quad (xy \text{ plane}) , \tag{7}$$

which ones are *continuous* across the boundary $z = 0$ and which ones are *discontinuous*? *Why?* What may be *physically present at the boundary* $z = 0$ that gives rise to these discontinuities?

e.) *(2 pts.)* Some of the components of $\vec{E}, \vec{B}$ are *continuous* across the boundary. Argue that for these continuity conditions to be valid for all times t and all positions $\vec{r} = (x, y, 0)$ on the boundary, the reflected and transmitted waves must satisfy

$$\omega = \omega' = \omega'' , \qquad \vec{k}_\perp = \vec{k}'_\perp = \vec{k}''_\perp , \tag{8}$$

where $\vec{k}_\perp = k_x \hat{i} + k_y \hat{j}$ denotes a vector in the xy-plane.

f.) *(1.5 pts.)* Show that Eqs. (8) imply both that the *angle of incidence equals the angle of reflection* and *Snell's law for the angle of refraction*:

$$\theta_i = \theta_r , \qquad n_1 \sin\theta_i = n_2 \sin\theta_t , \tag{9}$$

where $n = c/v$ is the index of refraction, v is the speed of the wave in the medium, and c is the speed of light. The angles are defined in Fig. 1.

g.) *(1 pt.)* We observe from Eqs. (9) that for $1 \geq \sin\theta_i \geq \frac{n_2}{n_1}$, the solution for θ_t (or equivalently, $\vec{k}'$) becomes *complex*. *What physically is occurring* to the electromagnetic waves when this happens?