Classical Mechanics

August 19, 2023

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

A bead subject to gravity is constrained to slide without friction on a parabolic wire described by the equation $y = bx^2$ with x being a horizontal axis and y being the vertical axis. Derive the equation of motion for the bead.

Problem 2

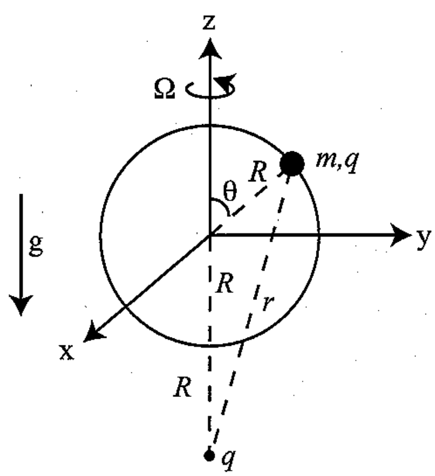
Under the influence of a central force originating at the origin of a polar coordinate system labeled by the radial coordinate r and the angular coordinate θ, a particle moves on an orbit described by

$$\frac{1}{r} = u_0 + u_1 \cos(3\theta/2) \ .$$

Find the potential generating this force.

Problem 3

A charged bead of mass m and electric charge q is threaded on a circular massless and frictionless hoop of radius R. The hoop rotates with angular velocity Ω around a vertical z axis and sits in a vertical gravitational field of acceleration g. A second, stationary charge q is placed along the z axis at a distance R below the bottom of the hoop. Use as generalized coordinate the angle θ between the bead and the vertical z axis (see figure).



a.) Using a Cartesian laboratory frame whose origin is at the center of the hoop, write down the transformation equations expressing the Cartesian coordinates x, y, z in terms of the generalized coordinate θ.

b.) Using the results from part a.), show that the kinetic energy of the bead is

$$T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) \ .$$

Also calculate its total potential energy as a function of θ. Recall that the electrostatic contribution has the form

$$V_{\text{elec}} = k \frac{q^2}{r} \quad (k > 0) \ ,$$

where $r = R\,(5 + 4\cos\theta)^{1/2}$ is the distance between the static charge and the bead.

(continued)

c.) Write down the Lagrangian of the bead and derive the corresponding Euler-Lagrange equation for θ.

d.) To simplify the problem, use the special value $\Omega^2 = \frac{4g}{5R}$. Show that there are always two "trivial" equilibrium positions $\theta_0 = 0, \pi$ for the bead (not necessarily stable) and that, for a certain range of the dimensionless ratio

$$\lambda = \frac{kq^2}{mgR^2} \ ,$$

there will be an additional non-trivial equilibrium position θ_0. Find the range of λ where this solution exists and express this new equilibrium position θ_0 in terms of λ. Finally, find the frequency of small oscillations of the system with respect to the non-trivial equilibrium position θ_0.

e.) Returning to the exact Lagrangian of part c.), calculate the Hamiltonian of the bead as a function of θ and its conjugate canonical momentum p_θ. Is this Hamiltonian a conserved quantity? Is it equal to the energy of the bead?