Electrodynamics

August 18, 2023

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Lifetime of a classical atom: Consider a classical atom made of a single electron initially orbiting circularly around a proton (assume the proton is a point charge with infinite mass, and the electron is also a point charge with mass m_e in this problem), with orbital radius equal to the Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/(m_e e^2)$.

a.) Without calculation, please comment on the fate of this classical atom and explain why.

b.) Assume the adiabatic approximation, where the orbit changing time is much longer than the orbital period; solve for the orbit radius as a function of time $r(t)$. *Hint: the Larmor formula for the total power radiated by a nonrelativistic point charge as it accelerates is* $P = \mu_0 q^2 a^2/(6\pi c)$.

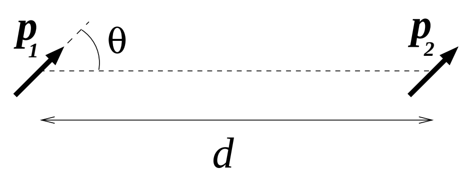
c.) What is the lifetime of this classical atom?

Problem 2

Two point electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ lie in the same plane a constant distance d apart. The dipole $\mathbf{p}_1$ makes a fixed angle θ with the line connecting the centers of the two dipoles, as shown in the figure below.

a.) Find the equilibrium angle of the dipole $\mathbf{p}_2$ to the line connecting the centers of $\mathbf{p}_1$ and $\mathbf{p}_2$.

b.) Calculate the total potential energy of this system of two dipoles in equilibrium.



Problem 3

Consider a Gaussian charge distribution with a different width transverse to the z-axis compared to the longitudinal width along the z-axis:

$$\rho(\vec{r}) = A \exp\left(-\frac{x^2}{2\sigma_T^2} - \frac{y^2}{2\sigma_T^2} - \frac{z^2}{2\sigma_L^2}\right) , \qquad \text{where } A = \frac{q_{\text{tot}}}{(2\pi)^{3/2}\sigma_T^2\sigma_L}$$

a.) For the case of the longitudinal width equal to twice the transverse width, $\sigma_L = 2\sigma_T$, determine the multipole moments q_{lm} and the electrostatic potential of this distribution up through $l = 2$.

b.) Assume now that the longitudinal width σ_L oscillates in time with the following form, $\sigma_L(t) = (2 - \cos(\omega t))\sigma_T$. What multipole moments are expected when $\omega t = 0$, and when $\omega t = \pi/2$? Qualitatively, discuss the pattern of radiation expected in those two cases ($\omega t = 0$ and $\omega t = \pi/2$).

Some possibly useful integration forms:

$$\int_0^\infty dx\, x^n e^{-ax^2} = \frac{(n-1)!!\sqrt{\pi}}{2^{(n/2)+1}} a^{-(n+1)/2} \quad \text{for even } n \geq 0$$

$$\int_{-1}^1 dx\, (4-3x^2)^{-1/2} = \frac{2\pi}{3\sqrt{3}} , \qquad \int_{-1}^1 dx\, x^2(4-3x^2)^{-1/2} = \frac{4\pi\sqrt{3}-9}{27}$$

$$\int_{-1}^1 dx\, (4-3x^2)^{-3/2} = \frac{1}{2} , \qquad \int_{-1}^1 dx\, x^2(4-3x^2)^{-3/2} = -\frac{2\pi\sqrt{3}-18}{27}$$

$$\int_{-1}^1 dx\, (4-3x^2)^{-5/2} = \frac{1}{4} , \qquad \int_{-1}^1 dx\, x^2(4-3x^2)^{-5/2} = \frac{1}{6}$$

$$\int_{-1}^1 dx\, (4-3x^2)^{-7/2} = \frac{3}{20} , \qquad \int_{-1}^1 dx\, x^2(4-3x^2)^{-7/2} = \frac{7}{60}$$

Problem 4

Consider a rectangular metallic waveguide, infinitely long in the z-direction, and of width (x-direction) a and height (y-direction) b. Let $a \geq b$. The lowest frequency wave that can propagate corresponds to a "transverse electric" mode for which the only non-zero electric field component is E_y.

a.) Assuming that the fields have (z, t) dependence $\exp[i(kz - \omega t)]$, show that the transverse fields for this mode are given in terms of B_z by

$$B_x = \frac{ik}{\gamma^2} \frac{\partial B_z}{\partial x} \quad \text{and} \quad E_y = -\frac{i\omega}{\gamma^2} \frac{\partial B_z}{\partial x} ,$$

where $\gamma^2 = (\omega/c)^2 - k^2$.

b.) Show that B_z satisfies the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \gamma^2 \right) B_z = 0 .$$

Also, what are the boundary conditions satisfied by B_z (assume that there are no currents flowing in the waveguide)?

c.) What is the minimum possible frequency of such a wave?

Problem 5

Properties of Gauge Fields:

a.) *(2 pts.)* You have designed a cavity to produce a static magnetic field of the form

$$\vec{B} = -2B_0 \left[(y+z)\hat{\imath} + (x-z)\hat{\jmath} + y\hat{k} \right] \tag{1}$$

with B_0 a constant, and your calculations give the corresponding magnetic vector potential to be

$$\vec{A} = B_0 \left[(x^2+y^2+z^2)\hat{\imath} + z^2\hat{\jmath} + (x^2-y^2+z^2)\hat{k} \right] \ . \tag{2}$$

A rival posts a paper with their own design for a cavity to produce the desired magnetic field (1), but in their calculations they use a different magnetic vector potential

$$\vec{A}' = B_0 \left[(2x^2+y^2)\hat{\imath} + 3y^2\hat{\jmath} + (x^2-y^2-2xz-2yz)\hat{k} \right] \ . \tag{3}$$

You are outraged that they would contradict you and begin to prepare a scathing rebuttal. **Is their result wrong, or is yours? Justify your answer.**

b.) *(3 pts.)* **Find a scalar function** $\Lambda(x,y,z)$ that performs the gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{4}$$

between the vector potentials (2) and (3).

c.) *(3 pts.)* Starting with Maxwell's equations in vacuum (in CGS units),

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{5}$$

prove that the vector potential $\vec{A}$ satisfies the wave equation

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \tag{6}$$

if we choose the **Coulomb gauge condition**

$$\vec{\nabla}\cdot\vec{A} = 0 \ , \quad \phi = 0 \tag{7}$$

where ϕ is the scalar potential.

(continued)

d.) *(2 pts.)* **Prove that the wave equation** (6) **is invariant** under the relativistic Lorentz boost

$$\begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix} \tag{8}$$

where η is a real constant parameter called the "boost rapidity."