Quantum Mechanics

August 17, 2023

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Consider a spinless charged particle in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. The Hamiltonian takes the form

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right)^2 ,$$

where e and m are the charge and mass of the particle, respectively, and $\vec{p} = (p_x, p_y, p_z)$ is the momentum conjugate to the particle's position $\vec{r}$. Let $\vec{A} = -By\hat{x}$, which corresponds to a constant magnetic field $B\hat{z}$ ($\hat{x}$ and $\hat{z}$ are unit vectors along the x and z axes, respectively.)

a.) Prove that p_x and p_z are constants of the motion.

b.) From a.) it follows that the corresponding eigenfunctions of H are of the form

$$\psi(x, y, z) = \exp[i(xp_x + zp_z)/\hbar]\phi(y) ,$$

where p_x and p_z are constants. Derive the eigenvalue equation satisfied by ϕ and determine the quantum energy levels. Provide a classical interpretation of the corresponding eigenstates.

Problem 2

Two electrons move in a central field. Consider the electrostatic interaction $V = e^2/|\vec{r}_1 - \vec{r}_2|$ between the two electrons as a perturbation.

a.) Construct the first order energy shifts for the two-electron states (terms) corresponding to the $1s^1 2s^1$ configuration. Express your answers in terms of unperturbed quantities and matrix elements of the interaction V.

b.) Discuss the symmetry of the two-particle wave functions for the states in part a.).

c.) Suppose that, at time $t = 0$, one electron is found to be in the $1s$ unperturbed state with spin up and the other electron is in the $2s$ unperturbed state with spin down. If the state evolves with time in the presence of the perturbation V, at what times will the 1s electron be in a spin-up state and the $2s$ electron in a spin-down state?

Problem 3

A one-dimensional harmonic oscillator is initially in the state $|\psi(0)\rangle = |1\rangle$. A time-dependent perturbation of the form

$$\hat{H}' = \left(\hat{a} + \hat{a}^\dagger\right) V_0 \, e^{-t/\tau}$$

acts on the system starting at $t = 0$, where $V_0 \ll \hbar\omega_0$, with ω_0 being the characteristic frequency of the oscillator.

a.) *(1 pt.)* Write down the complete Hamiltonian of the system, including the perturbation.

b.) *(3 pts.)* Working in the interaction picture, derive $|\psi_I(t)\rangle$ to first order in $\lambda \equiv V_0/\hbar\omega_0$.

c.) *(3 pts.)* From the result in b.), derive the state $|\psi_S(t)\rangle$ in the Schrödinger picture.

d.) *(3 pts.)* Calculate the probability $P_{1\to n}(t)$ for a transition to the n state to occur after time t, due to the perturbation.

Problem 4

Consider a particle of charge e and mass m moving on a two-dimensional plane under the influence of a scalar potential $V(\vec{r}) = \alpha x$ and a vector potential $\vec{A} = xB\hat{y}$ (assume units $e = c = 1$).

a.) First consider a special case where $\alpha = p_y B/m$, and determine the energy levels of the system.

b.) Now consider a more generic case where α can be any value and is independent of p, B, and m. How do the energy levels differ from the special case above? Draw a sketch of the expected trajectory of a particle in the classical limit as it moves through the x-y plane.

c.) In the limit $\alpha \to 0$, are both p_x and p_y conserved? If not, construct modified quantities p'_x and/or p'_y that are conserved.

d.) Determine the mean velocity of a particle traveling in the y-direction.

Problem 5

A cavity atom: Consider a "cavity atom" made of a single electron trapped in a spherical cavity with radius equal to the Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/(m_e e^2)$, where the potential is

$$V(\mathbf{r}) = \begin{cases} 0 & \text{for} \quad r \leq a_0 \\ \infty & \text{for} \quad r > a_0 \end{cases}$$

The cavity wall is made of an ideal conductor, so that electromagnetic (EM) waves, if there are any, are also trapped inside the cavity.

a.) Use the method of separation of variables for the single electron Schrödinger equation; write down the angular and radial parts of the equation.

b.) Solve the above equations and write down the solutions for the wavefunctions in the form $\psi = \psi(r, \theta, \phi)$.

c.) Solve for the first five lowest energy levels in units of eV, ordered from lower to higher, and also use the labels of a hydrogen atom (i.e., 2s, 2p, 3d, ...) to label these energy levels.

d.) Without calculation, what is the half-life $t_{1/2}$ of the first excited state, i.e., the time it takes an electron in the first excited state to have 50% chance of having fallen into the ground state? In terms of this $t_{1/2}$, what is the essential difference between this cavity atom and a hydrogen atom?

Hint: The following differential equation for $R_l(r)$,

$$\frac{d^2 R_l}{dr^2} + \frac{2}{r}\frac{dR_l}{dr} + \left[k^2 - \frac{l(l+1)}{r^2}\right] R_l = 0$$

has two linearly independent solutions:

$$j_l(x) = x^l \left(-\frac{1}{x}\frac{d}{dx} \right)^l \left(\frac{\sin x}{x} \right)$$

$$n_l(x) = -x^l \left(-\frac{1}{x}\frac{d}{dx} \right)^l \left(\frac{\cos x}{x} \right)$$

where $x = kr$, and these solutions are known as the spherical Bessel functions (converge at $x = 0$) and von Neumann functions (diverge at $x = 0$), respectively.

(continued)

The first zeroes of $j_l(x)$ and $n_l(x)$ are as follows:

$j_l(x)$	1st	2nd	3rd	4th
$l = 0$	3.142	6.283	9.425	12.566
$l = 1$	4.493	7.725	10.904	14.066
$l = 2$	5.763	9.905	12.323	15.515
$l = 3$	6.988	10.417	13.698	16.924
$l = 4$	8.813	11.705	15.040	18.301

$n_l(x)$	1st	2nd	3rd	4th
$l = 0$	1.571	4.712	7.854	10.996
$l = 1$	2.798	6.121	9.318	12.487
$l = 2$	3.959	7.452	10.716	13.928
$l = 3$	5.088	8.734	12.068	15.315
$l = 4$	6.198	9.982	13.385	16.677