Quantum Mechanics

Do <u>*two*</u> *of the following three problems, each on a separate page (or pages) and write your name on every page you turn in.*

Problem 1

A tritium atom (^{3}H, with one proton and two neutrons in its nucleus) decays into a ^{3}He (helium-3) atom, whose nucleus contains two protons and one neutron. The transformation can be considered as instantaneous. If before the decay the ^{3}H atom was in its ground state, what is the probability that the newly-created ^{3}He atom will be in its ground state? The radial wavefunctions of the ground states of hydrogen-like atoms are given by

$$R_{1,0}(r) = \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} 2e^{-Zr/a_0},$$

where $a_0 \approx 0.05\,\text{nm}$ is the Bohr radius and Z the atomic number.

Problem 2

A particle of mass m moves (non-relativistically) in the three-dimensional potential

$$V = \frac{1}{2}k\left(x^2 + y^2 + z^2 + \varepsilon xy\right),$$

where x, y, and z are the three spacial coordinates, k is the spring constant, and ε is a small real number; $\omega \equiv \sqrt{k/m}$.

(a) For the raising and lowering operators $\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$ and $\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$, where $\hat{x}$ and $\hat{p}_x$ are the position and linear-momentum operator, respectively, show the following <u>*three*</u> relationships:

$$\begin{aligned} \left[\hat{a}, \hat{a}^\dagger\right] &= 1 \\ \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\,. \end{aligned}$$

(b) Determine the energy of the ground state up to and including 2$^{\text{nd}}$ order perturbation theory.

(c) Determine the effect of the perturbation on the first excited state of the system. Is the degeneracy of the system lifted?

Problem 3

Suppose you have two *identical* non-interacting spin-$1/2$ particles in a one-dimensional harmonic oscillator potential of angular frequency ω. Let $\phi_n(x_i)$ denote the normalized position wave function for the n^{th} harmonic oscillator eigenstate of the i^{th} particle. ($n = 0, 1, 2, ...$ and $i = 1, 2$) This two-particle system is in an energy eigenstate with eigenvalue $3\hbar\omega$. If the z-component of the total spin of the two particles ($S_z = S_{1,z} + S_{2,z}$) is zero, write down all possible normalized total wave functions (as a product of spatial and spin parts) for the system.