Modern Physics

Do two of the following three problems, each on a separate page (or pages) and write your name on every page you turn in.

Problem 1

Consider the motion of a one-dimensional particle of mass m in the potential

$$V(x) = V_0 \left(\frac{x}{a}\right)^{2n},$$

where n is a positive integer, a is a real-valued, positive constant, and $V_0 > 0$.

(a) Discuss the parity of the eigenfunctions.

(b) Use the uncertainty principle to estimate the lowest-energy eigenvalue.

(c) Specialize this estimate to $n = 1$ and $n \to \infty$. Clearly state what $V(x)$ becomes in each case and compare to the exact solutions.

Problem 2

Consider two electrons, one in an infinite square well potential and one in a harmonic oscillator potential, both in one dimension. Both electrons are in the lowest-energy state. The longest-wavelength photon that these electrons can absorb has wavelength $\lambda = 1000\,\text{Å}$.

(a) Find the width of the square well, in Å.

(b) Find the classical amplitude of oscillation for the electron in the ground state of this harmonic oscillator potential, in Å.

(c) Compare the probabilities that the electron is at the center of the well in each case. Which probability is larger?

(d) Draw the wavefunctions for both cases and point out the similarities and differences.

Hints: $\int_{-\infty}^{\infty} e^{-\eta x^2} dx = \sqrt{\frac{\pi}{\eta}}$; $\quad \frac{\hbar}{m_e} = 7.62\,\text{eV}\,\text{Å}$.

Problem 3

An electron gun produces electrons randomly polarized with spins up or down along one of three possible, randomly selected, orthogonal axes 1, 2, 3 (i.e., x, y, z), with probabilities $p_{i,\uparrow}$ and $p_{i,\downarrow}$, $i = 1, 2, 3$. To simplify the final results, it is better to rewrite these in terms of d_i and δ_i defined by

$$p_{i,\uparrow} = \frac{1}{2}d_i + \frac{1}{2}\delta_i, \qquad p_{i,\downarrow} = \frac{1}{2}d_i - \frac{1}{2}\delta_i, \qquad i = 1,\, 2,\, 3.$$

Probabilities must be non-negative, so $d_i \geq 0$ and $|\delta_i| \leq d_i$.

(a) Write down the resultant electron spin density matrix ρ in the basis $|\uparrow\rangle$, $|\downarrow\rangle$ with respect to the z-axis.

(b) Any 2×2 matrix ρ can be written as

$$\rho = a\mathbf{1} + \mathbf{b} \cdot \sigma$$

in terms of the unit matrix and 3 Pauli matrices. Determine a and $\mathbf{b}$ for ρ from part (a).

(c) A second electron gun produces electrons with spins up or down along a single axis in the direction $\hat{\mathbf{n}}$ with probabilities $(1 \pm \Delta)/2$. Find $\hat{\mathbf{n}}$ and Δ so that the electron ensemble produced by the second gun is the same as that produced by the first gun.