Modern Physics

Do two of the following three problems, each on a separate sheet (or sheets). Attach each set to a provided cover sheet with your name, subject, and problem number.

Problem 1

The Hamiltonian of a free rigid rotor is given by

$$\hat{H} = \frac{\hat{L}_z^2}{2I} ,$$

where $\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$ is the z-component of the orbital angular momentum operator and I the moment of inertia.

a. Show that the functions

$$\psi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{im\varphi},\ \ m = 0, \pm1, \pm2, \dots ,\ \ \varphi \in [0, 2\pi],$$

are simultaneous eigenfunctions of $\hat{H}$ and $\hat{L}_z$ in position space. Determine the corresponding eigenvalues. (3 points)

b. Determine the time evolution of a state that has the following form,

$$\psi(\varphi, t = 0) = A\,\sin^2\varphi,$$

at an initial time $t = 0$. At what later times does the rotor momentarily return to its initial state? (7 points)
Hint: Expand the initial state in terms of eigenfunctions $\psi_m(\varphi)$.

Problem 2

Consider a particle of mass m and energy $E < 0$ in a one-dimentional, δ-function potential $V(x) = a\delta(x)$, where $a < 0$ is a constant.

a. Starting from the time-independent Schrödinger equation, derive the continuity conditions for the wavefunction $\psi(x)$ and its first derivative at $x = 0$. (4 points)

b. Solve the Schrödinger equation to show that there exists only one bound state and obtain the wavefunction and energy of that state. (6 points)

Problem 3

In this problem you will estimate the depth of a potential well for a deuterium nucleus with a ground-state energy $E_0 = -2.2$ MeV. For simplicity we assume a model potential of the form $V(r) = -a\exp(-r/a)$ with the unknown well depth A and an interaction range $a = 2.2$ fm. Moreover, we choose a trial wavefunction with one adjustable parameter α: $\psi(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi)$, where $Y_{lm}(\theta,\phi)$ is a suitable spherical harmonic and $R(r) = c\exp(-\alpha r/a)$.

a. Prove Ritz's variational principle for the ground-state energy E_0:

$$E_0 \leq \frac{\left\langle \psi \middle| \hat{H} \middle| \psi \right\rangle}{\langle \psi | \psi \rangle}$$

for arbitrary wavefunction ψ and a given Hamiltonian $\hat{H}$.

b. In our 3-dimensional, spherically symmetric problem, what can you say about the angular dependence of the ground-state wavefunction? What is the corresponding normalized spherical harmonic? Explain.

c. Using energy arguments, evaluate if we should expect to find a bound state for the deuteron in our model potential. Explain.

d. After performing the integrals in Ritz's variational principle, we obtain the following expression (no need for you to do the integrals):

$$E_0 \leq \frac{\hbar^2}{2\mu}\frac{\alpha^2}{4a^2} - A\frac{\alpha^3}{(1+\alpha)^3}.$$

Estimate the depth A of the potential for the experimentally known binding energy of the deuteron in the ground state, $E_0 = -2.2$ MeV and an interaction range $a = 2.2$ fm. Show and explain your math.

Hints:

1. The radial part of the 3-dimensional Schrödinger equation is

$$\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)R(r) + \frac{\hbar^2 l(l+1)}{2m_0 r^2}R(r) + V(r)R(r) = ER(r).$$

2. Constants: $h = 6.626 \times 10^{-34}$ J s; 1 amu $= 1.661 \times 10^{-27}$ kg.