Quantum Mechanics

Do two of the following three problems, each on a separate sheet (or sheets). Staple together the sheets for each problem, if using multiple sheets, but do not staple all problems together. Write at the top of the first sheet of each problem your name, subject, and problem number.

Problem 1

(a) *(2 pts.)* Show that the eigenfunctions $\phi_k(x)$ of the momentum operator $\hat{p} \to -\imath\hbar\frac{d}{dx}$ are plane waves

$$\phi_k(x) = Ae^{ikx} \tag{1}$$

by directly solving the eigenvalue equation

$$\hat{p}\phi_k(x) = \hbar k \phi_k(x). \tag{2}$$

(b) *(2 pts.)* By comparing with the Taylor series, show that the translation operator $e^{-\imath\hat{p}a/\hbar}$ shifts the spatial argument of any wave function ψ by the displacement a:

$$e^{-\imath\hat{p}a/\hbar}\psi(x) = \psi(x-a). \tag{3}$$

(c) *(2 pts.)* Independently of Part (a), use the translation operator (3) to show that the eigenfunctions (2) of the momentum operator are plane waves.
Hint: Consider translations of the wave function from $x = 0$.

(d) *(4 pts.)* The probability current j is related to the probability density $P(x,t) = |\psi(x,t)|^2$ by

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x}. \tag{4}$$

Calculate the probability current j carried by the momentum eigenfunction (1).
Hint: Use the Schrödinger equation to derive j from (4).

Problem 2

A particle in a central potential has an orbital angular momentum $l = 1\hbar$ and a spin $s = 1\hbar$. Find the energy levels and degeneracies associated with a spin-orbit interaction term of the form $H_{\rm SO} = A\vec{L}\cdot\vec{S}$, where A is a constant and $\vec{L}$, $\vec{S}$ are the usual angular momentum vector and the vector consisting of the Pauli spin matrices, respectively.

Problem 3

Consider a particle of mass m in a one-dimensional, δ-function potential

$$V(x) = \lambda\delta(x),$$

where λ can be positive or negative.

(a) *(4 points)* Write the time-independent wavefunction $\psi(x)$ taking into account all the relations between coefficients imposed by the boundary and normalization conditions.

(b) *(3 points)* Show that there exists a bound state for a δ well but not for a δ barrier.

(c) *(3 points)* Find the energy of the bound state in terms of λ and m.