Quantum Mechanics

Do two of the following three problems, each on a separate sheet (or sheets). Staple together the sheets for each problem, if using multiple sheets, but do not staple all problems together. Write at the top of the first sheet of each problem your name, subject, and problem number.

Problem 1

Consider four Hermitian 2×2 matrices I, σ_1, σ_2, and σ_3, where I is the identity matrix, and the other three satisfy

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}.$$

Prove the following statements without using a specific representation or form for the matrices:

a) Prove that $\mathrm{Tr}(\sigma_i) = 0$, where $\mathrm{Tr}(A) = \sum_{i=1}^{2} A_{ii}$ is the sum of the diagonal elements (trace of the matrix).

b) Show that the eigenvalues of σ_i are ± 1 and that $\det(\sigma_i) = -1$.

c) Show that the four matrices are linearly independent and explain what this means.

Hints: You may use, without proving:

- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
- $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

Problem 2

A particle is described by the wave function

$$\psi(x,t) = A_0 \psi_0(x) e^{-\frac{i}{\hbar} E_0 t} + A_1 \psi_1(x) e^{-\frac{i}{\hbar} E_1 t},$$

where $\psi_i(x)$ are normalized solutions to the stationary Schrödinger equation with energy E_i ($i = 0,\ 1$). A_0 and A_1 are contants.

a) Derive an equation for A_0 and A_1 so that $\psi(x,t)$ is properly normalized.

b) At time t_1 the energy is measured. Calculate the probability, as a function of t_1, that the measured energy is E_1.

c) At time $t_2 > t_1$ the energy is measured again. Calculate the probability that the energy is E_0.

Show all the steps and explain, rather than simply stating the results.

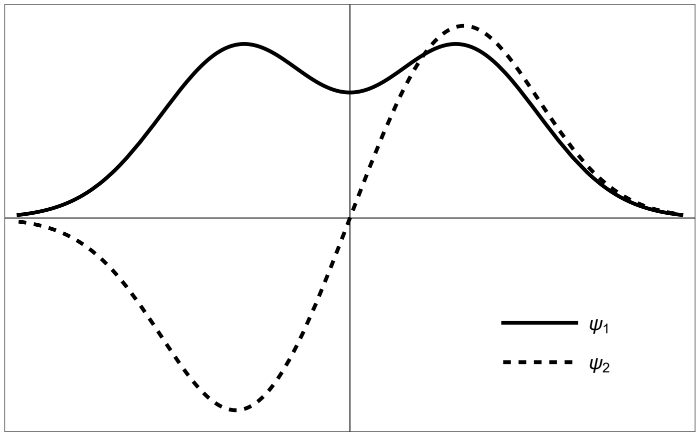

Figure 1: Two wave functions $\psi_1(x)$ and $\psi_2(x)$

Problem 3

a) (*3 pts.*) A quantum system has a Hamiltonian of the form

$$\mathcal{H} = \begin{bmatrix} \varepsilon & \Delta_1 & 0 \\ \Delta_1 & \varepsilon & \Delta_2 \\ 0 & \Delta_2 & \varepsilon \end{bmatrix}, \tag{1}$$

where ε, Δ_1, and Δ_2 are real constants. **What are all possible outcomes** of measuring the energy using the Hamiltonian of Eq. 1?

b) (*3 pts.*) Suppose the angular momentum operator L_z of a quantum system is given by the matrix

$$\hat{L}_z = \begin{bmatrix} 0 & \hbar & 0 \\ \hbar & 2\hbar & \hbar \\ 0 & \hbar & 0 \end{bmatrix},$$

where $\hbar$ is the reduced Planck constant. If a measurement of $\hat{L}_z$ yields the result $\hat{L}_z = 0$, **what properly normalized state** will the system be in afterwards?

c) (*2 pts.*) Two wave functions $\psi_1(x)$ and $\psi_2(x)$ are plotted in Fig. 1 above: Which of the two wave functions $\psi_1(x)$ and $\psi_2(x)$ has **lower kinetic energy**? **Justify your answer**.

d) (*2 pts.*) A quantum system consists of two non-interacting particles, one with spin $s_1 = \frac{3}{2}$ and the other with $s_2 = \frac{1}{2}$. **List all possible outcomes** of a measurement which yields the **total** angular momentum j and its z-component j_z.