Thermodynamics

Do two of the following three problems, each on a separate sheet (or sheets). Attach each set to a provided cover sheet with your name, subject, and problem number.

Problem 1

Consider a system of $n = 4$ atoms, each of which carries an equal spin that could be up ($\uparrow$) or down ($\downarrow$).

(a) Write down symbolically with arrows up and down all the possible arrangements of the four spins. (1 point)

(b) What is the total number of different arrangements of the four spins, i.e., the number of microstates? (2 points)

(c) When a magnetic field is applied, the energy of the system depends on the total net spin,

$$U = -2smB$$

where $2s$ is the excess spin $2s = n_{\uparrow} - n_{\downarrow}$, m is the magnetic moment of the individual spin, and B is the magnetic field. Make a sketch of the energy diagram showing the number of energy levels (macrostates) and the arrangements of spins that correspond to each level. How many arrangements of the spins correspond to each of the energy levels (the multiplicity, or degeneracy, of the levels)? (3 points)

(d) Rather than simply counting, finding the multiplicity of each level is also possible using the appropriate binomial coefficients

$$\left(\frac{n!}{(n-r)!r!} \right).$$

You may assume here that r is the number of spins in spin-up configuration. Check whether the results are consistent with part **(c)**. (1 point)

(e) In most statistical-physics problems, however, one needs to use an approximate formula for the multiplicity function, given by

$$\Omega(n, r) \approx \sqrt{\frac{2}{n\pi}} 2^n \exp\left(-\frac{2s^2}{n} \right),$$

where $s \equiv r - \frac{n}{2}$. Use this approximation to estimate the multiplicity of each energy level for the above system and compare with your results in part **(d)**. Compute the percent difference. You will find significant deviations. Why? (Note: The derivation of this approximation relies on the Stirling approximation and a Taylor expansion of the logarithm.) (3 points)

Problem 2

Consider a *two-dimensional* monatomic classical ideal gas composed of particles of mass m and held at temperature T. Find the average speed $\langle v \rangle$, the root-mean-square speed $\sqrt{\langle v^2 \rangle}$, and the most probable speed of the gas particles.

Problem 3

We know that an ideal gas cools down during an adiabatic expansion. On the other hand, a one-dimensional rubber band is increasing its temperature when elongated (stretched) adiabatically.

(a) Write down the first law of thermodynamics for the elongation of a rubber band with spring constant k, looking at possible similarities with the ideal gas, and justify the above statement; pay attention to the appropriate signs. (5 points)

(b) What happens to the entropy if the rubber band is stretched isothermally? Use Maxwell-type relations derived from an appropriate thermodynamic potential. (5 points)